

A Bivariate Generalized Inverse Weibull Distribution for Advanced Modeling in Economics and Environmental Science

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Abstract:

Introduction: Modeling economic and environmental phenomena is a great importance due to the complex, uncertain and independent nature of real-world data arising in these fields. This paper introduces a new bivariate distribution known as FGM bivariate Generalized Inverse Weibull distribution. The proposed model is constructed by combining Generalized Inverse Weibull marginals through FGM copula framework, allowing for flexible modelling of dependence in economic and environmental data.

Methodology: Some of the important statistical properties of the proposed distribution are discussed which include the following: conditional distributions, conditional expectations, product moment, moment generating functions, and reliability functions. The estimation of the distribution parameters is carried out using maximum likelihood estimation method, inference functions for margins and Bayesian estimation method. Interval estimation is provided through confidence intervals for MLE and IFM and higher posterior density intervals for Bayesian estimates. To assess, the performance of the estimator in terms of bias, mean square error and interval length a simulation study is conducted.

Results: The results of the simulation show that all methods with improved accuracy for larger sample sizes. The utility of the developed distribution is discussed using empirical data from both economic and environmental science, showing greater flexibility and fit than the existing models.

Conclusion: The developed distribution offers a better model for modeling complicated bivariate relationships in economics and environmental science.

Keywords: Generalized inverse weibull distribution, farlie-gumbel morgenstern copula, dependence modeling, reliability analysis, parameter estimation, simulation study.

1. INTRODUCTION

In many real-world situations, when two or more random variables are observed and show some degree of dependence, the joint behavior of such variables is not described using univariate probability distributions alone. On the other hand, bivariate probability distributions give a mathematical way of studying the joint distribution of two or more random variables. Bivariate probability distributions have many applications in

various fields such as economics, engineering, reliability theory, biomedical sciences, and environmental sciences.

The bivariate probability distribution gives the probability distribution of two random variables through their joint probability density function or mass function. It takes into account not only the probability distributions of the variables individually but also their association. There are many features of the distribution that can be derived from the joint probability

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distribution including the correlation coefficient, marginal distributions, conditional distributions, among others. Several bivariate probability distributions have been developed to model the joint behavior of two random variables *i.e.*, Yoo *et al.* [1], Herr *et al.* [2], Olkin *et al.* [3], Yue *et al.* [4], Burgazzi *et al.* [5], Kocherlakota *et al.* [6], Yue [7], Li *et al.* [8], Yue *et al.* [9], Xie *et al.* [10], Nadarajah [11], Kim *et al.* [12], Lai *et al.* [13], Dehghani *et al.* [14], Filus *et al.* [15], Castillo *et al.* [16], Alaya *et al.* [17], Erdem *et al.* [18], Wang *et al.* [19], Onyancha *et al.* [20], Jacobs [21], Lee *et al.* [22], Poonia *et al.* [23], Patra *et al.* [24] and Ding [25] among others.

There exist various methods to create bivariate probability distribution in the literature, and copula is one of them. Nelsen [26] describes the copula as a function which joins the bivariate distribution functions having uniform marginals on the range [0,1]. The dependence relationship between random variables can be labeled by a copula using their quantile behavior. By combining univariate marginal distributions with an appropriate copula to construct bivariate probability distributions. There are numerous types of copulas including Clayton copula, Gumbel copula, t copula, Gaussian copula, Farlie-Gumbel-Morgenstern (FGM) copula, Archimedean copula and Frank copula. Among the various copulas, the Farlie-Gumbel-Morgenstern (FGM) copula is widely used for constructing bivariate probability distribution. For example, Shih *et al.* [27], Cuadras *et al.* [28], Hussein *et al.* [29], El-Sherpieny *et al.* [30], Suzuki *et al.* [31], Louzada *et al.* [32], Abd Elgawad *et al.* [33], Qura *et al.* [34], Ghosh [35], El-Sherpieny *et al.* [36], Barakat *et al.* [37], Javed *et al.* [38], Ahmed *et al.* [39], Cuadras *et al.* [40], Hassan *et al.* [41] and Muhammed *et al.* [42] among others.

Although several bivariate lifetime distributions have been proposed in the literature, many of them suffer from one or more limitations. Some models have complicated joint structures that make parameter estimation computationally demanding, while others lack analytical tractability for deriving important reliability measures. Additionally, very few research articles have attempted to construct bivariate distributions using the Generalized Inverse Weibull distribution under the assumption of copula theory, specifically the Farlie-Gumbel-Morgenstern (FGM) copula. The reason is that the FGM copula offers an easy way to model weakly correlated variables. Hence, there exists a need for a flexible yet tractable bivariate Generalized Inverse Weibull distribution that can handle dependent lifetime data.

In this paper, we propose a new bivariate probability distribution, termed the FGM bivariate Generalized Inverse Weibull (FGM-GIW) distribution, constructed using the Farlie Gumbel Morgenstern (FGM) copula to link two Generalized Inverse Weibull marginal distributions. The proposed model is motivated by the need for a joint distributional framework that combines the flexible marginal behavior of the GIW distribution with an explicit dependence structure for modeling paired positive-valued data. By integrating these components, the model provides a framework for investigating the joint characteristics of two dependent random variables while retaining the individual properties of the GIW marginals. The

statistical properties of the proposed distribution are investigated, including conditional distributions, conditional expectations, product moments, moment-generating functions, and reliability functions. Parameter estimation is performed using Maximum Likelihood Estimation (MLE), the Inference Functions for Margins (IFM) method, and Bayesian estimation. The performance of the proposed model is evaluated through numerical simulations and applications to real datasets, demonstrating its potential utility in the analysis of dependent economic and environmental observations.

The FGM copula was considered because it is an easy-to-work-with structure that allows one to construct bivariate distributions while preserving all the properties of the marginal distributions of the Generalized Inverse Weibull distribution. The advantage of the mathematical simplicity of this model is that various theoretical characteristics of the distribution can be obtained in an explicit form, including conditional distributions, moments, reliability functions, and estimation methods for parameters of the distribution. However, it should be emphasized that the dependence range of the FGM copula is quite narrow and it is impossible to model strong dependence or tail dependence with the help of this structure. Therefore, the suggested FGM-GIW distribution is well suited for problems related to modeling weak to moderate dependence structures.

The remainder of the paper is organized as follows. In Section 2, the proposed bivariate FGM Generalized Inverse Weibull (GIW) distribution is introduced. Section 3 presents its statistical properties, while Sections 4 and 5 discuss reliability measures and parameter estimation methods, respectively. Section 6 provides the results of simulation studies, Section 7 illustrates applications to real datasets, and Section 8 concludes the paper.

2. MODEL DESCRIPTION

The Probability Density Function (PDF) and Cumulative Distribution Function (CDF) for the univariate Generalized Inverse Weibull distribution is given as Eq: (2.1).

$$F(x) = e^{-\gamma\left(\frac{\alpha}{x}\right)^\beta} \quad x > 0, \alpha, \beta, \gamma > 0 \quad (2.1)$$

and the corresponding PDF is shown in Eq: (2.2).

$$f(x) = \gamma\beta\alpha^\beta x^{-\beta-1} e^{-\gamma\left(\frac{\alpha}{x}\right)^\beta} \quad x > 0, \alpha, \beta, \gamma > 0 \quad (2.2)$$

where α, γ are the scale parameters and β is the shape parameter.

Fig. (1) presents the plots for PDF and CDF of univariate Generalized Inverse Weibull distribution under different parameter setting.

Let two random variables X and Y with marginal cumulative distribution functions $F(x)$ and $F(y)$ respectively. Then Sklar [43] presents the joint PDF and CDF based on copula theory Eq: (2.3).

$$F(x, y) = C[F(x), F(y)] \quad (2.3)$$

and the corresponding joint PDF is Eq: (2.4).

$$f(x, y) = f(x)f(y)c[F(x), F(y)] \quad (2.4)$$

The FGM copula's joint CDF can be expressed as Eq: (2.5).

$$C(x, y) = F(x)F(y)[1 + \theta\{1 - F(x)\}\{1 - F(y)\}] \quad (2.5)$$

and the corresponding joint PDF is Eq: (2.6).

$$c(x, y) = f(x)f(y)[1 + \theta\{1 - 2F(x)\}\{1 - 2F(y)\}] \quad (2.6)$$

where $-1 < \theta < 1$. The CDF of bivariate FGM Generalized Inverse Weibull (GIW) distribution is given as Eq: (2.7).

$$F(x, y) = e^{-\gamma_1(\frac{\alpha_1}{x})^{\beta_1}} e^{-\gamma_2(\frac{\alpha_2}{y})^{\beta_2}} \left[1 + \theta \left\{ 1 - e^{-\gamma_1(\frac{\alpha_1}{x})^{\beta_1}} \right\} \left\{ 1 - e^{-\gamma_2(\frac{\alpha_2}{y})^{\beta_2}} \right\} \right] \quad (2.7)$$

where $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2 > 0, -1 < \theta < 1$ and $x, y > 0$. The corresponding joint PDF is Eq: (2.8).

$$f(x, y) = \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1(\frac{\alpha_1}{x})^{\beta_1}} \right] \left[\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2(\frac{\alpha_2}{y})^{\beta_2}} \right] \left[1 + \theta \left\{ 1 - 2e^{-\gamma_1(\frac{\alpha_1}{x})^{\beta_1}} \right\} \left\{ 1 - 2e^{-\gamma_2(\frac{\alpha_2}{y})^{\beta_2}} \right\} \right] \quad (2.8)$$

Figs. (2 and 3) presents the plots for PDF and CDF of bivariate FGM-GIW distribution under different parameter setting.

The Table 1 presents the descriptive statistics of the bivariate FGM-GIW distribution for varying values of the parameter θ , ranging from -1 to 1. As θ increases, the mean of the distribution decreases indicating a gradual shift of the

central tendency towards lower values. Simultaneously, the variance decreases showing that the distribution becomes more concentrated around the mean. Skewness is on the increase, which indicates positive skewing. The same applies to kurtosis, which indicates that the data is highly leptokurtic. Covariance and correlation also indicate slight reduction in the relationship between the variables.

Table 1. Overview of distributional statistics across different parameter settings.

Parameters	$\alpha_1 = 2.0, \beta_1 = 3.0, \gamma_1 = 1.5, \alpha_2 = 1.5, \beta_2 = 4.0, \gamma_2 = 2.0$					
θ	Mean	Variance	Skewness	Kurtosis	Covariance	Correlation
-1	0.0581	0.0643	5.1234	38.2415	0.0312	0.5981
-0.8	0.0537	0.0615	5.2412	39.5123	0.0297	0.5945
-0.6	0.0508	0.0589	5.3581	40.7241	0.0281	0.5912
-0.4	0.0482	0.0564	5.4721	42.1035	0.0267	0.5880
-0.2	0.0457	0.0538	5.5892	43.4123	0.0254	0.5848
0	0.0431	0.0512	5.7125	44.8241	0.0240	0.5815
0.2	0.0405	0.0487	5.8412	46.2154	0.0227	0.5782
0.4	0.0382	0.0465	5.9621	47.6142	0.0213	0.5750
0.6	0.0361	0.0441	6.0835	49.0215	0.0200	0.5718
0.8	0.0340	0.0418	6.2041	50.4125	0.0187	0.5685
1	0.0320	0.0395	6.3214	51.8243	0.0175	0.5652

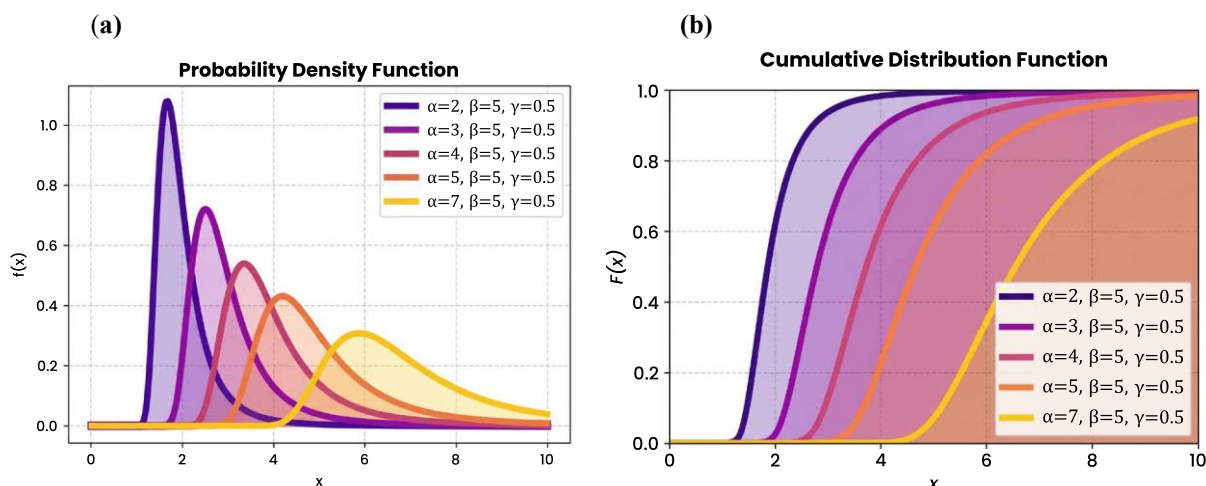


Fig. (1). Visual illustration of the Generalized Inverse Weibull distribution.

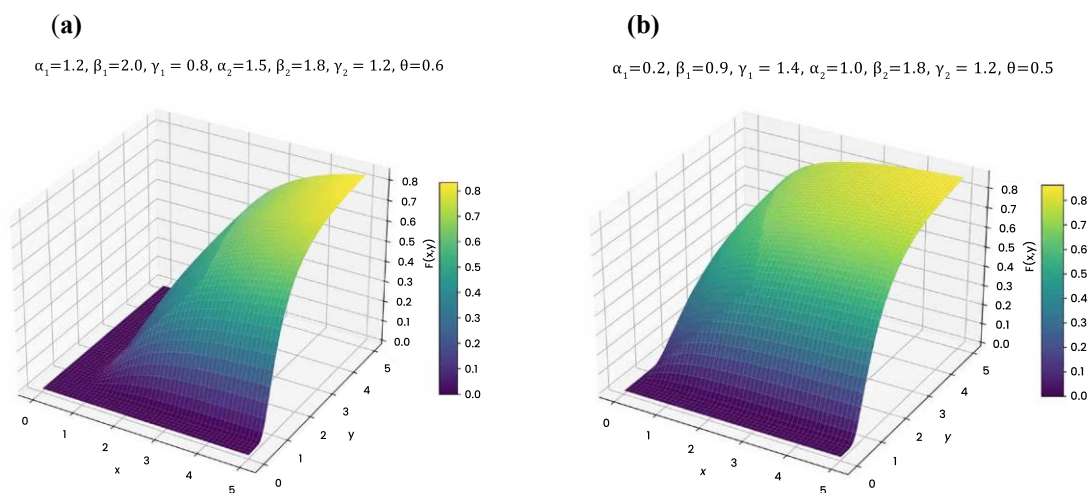


Fig. (2). Visualization of the CDF of bivariate FGM-GIW distribution.

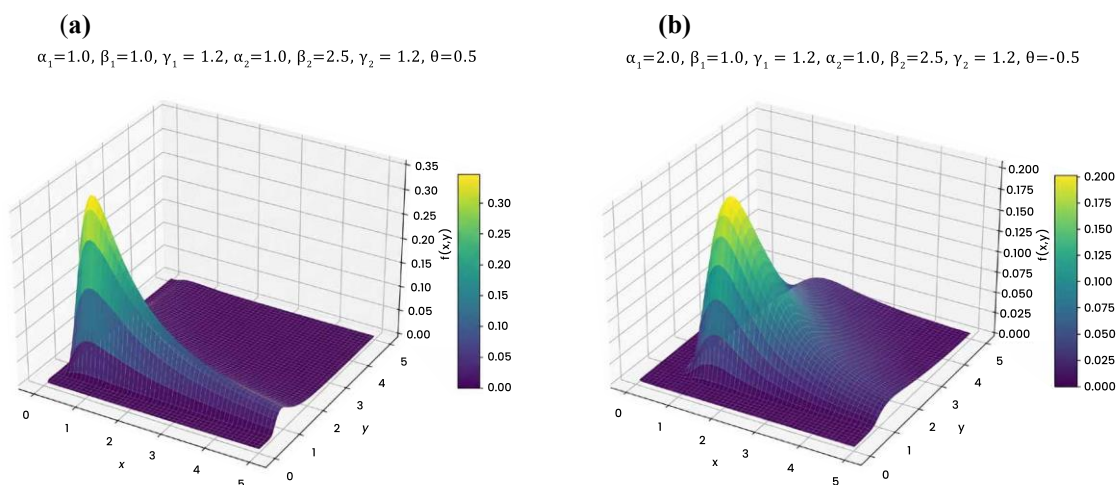


Fig. (3). Visualization of the PDF of bivariate FGM-GIW distribution.

3. STATISTICAL PROPERTIES

This section focuses on the derivation of key statistical properties of bivariate FGM-GIW distribution including validity tests, marginal distributions, conditional distributions, conditional expectations, method for random number generation, product moment and moment generating function.

3.1. Validity Test

To verify that the PDF defined in equation (2.8) is valid probability distribution, we check the fundamental condition that the integral of the density function over its entire support equals to one $\int_{-\infty}^{\infty} f(x, y) dy dx = 1$. To conform this property, we integrate equation (2.8) as follow:

$$\begin{aligned} & \int_0^{\infty} \int_0^{\infty} f(x, y) dy dx = 1 \\ &= \int_0^{\infty} \int_0^{\infty} \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left[\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right] \left[1 + \theta \left\{ 1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right\} \left\{ 1 - 2e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right\} \right] dy dx \\ &= \int_0^{\infty} \int_0^{\infty} \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left[\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right] dy dx + \theta \int_0^{\infty} \int_0^{\infty} \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left[\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right] \\ & \quad \left\{ 1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right\} \left\{ 1 - 2e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right\} dy dx \end{aligned}$$

Now solve each integral separately. Take the first term of the integral.

$$\begin{aligned} &= \int_0^{\infty} \int_0^{\infty} \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left[\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right] dy dx \\ &= \int_0^{\infty} \left\{ \int_0^{\infty} \left[\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right] dy \right\} \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] dx \end{aligned}$$

To solve the integral, we use substitution method. Let,

$$u = \gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}, du = -\gamma_2 \alpha_2^{\beta_2} \beta_2 y^{-\beta_2-1} dy, -du = \gamma_2 \alpha_2^{\beta_2} \beta_2 y^{-\beta_2-1} dy$$

The new ranges for $\infty \leq u \leq 0$.

$$\begin{aligned} &= \int_0^{\infty} \left\{ \int_{\infty}^0 [e^{-u} - du] dy \right\} \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] dx \\ &= \int_0^{\infty} \left\{ \int_0^{\infty} [e^{-u} du] dy \right\} \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] dx \\ &= \int_0^{\infty} [1] \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] dx \\ &= \int_0^{\infty} \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] dx \end{aligned}$$

The same substitution is used and then integrating with respect to x .

$$= \int_0^{\infty} \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] dx = 1$$

$$= \int_0^\infty \int_0^\infty \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left[\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right] dy dx = 1$$

Now the second term of the integral.

$$\begin{aligned} &= \theta \int_0^\infty \int_0^\infty \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left[\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right] \left\{ 1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right\} \left\{ 1 - 2e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right\} dy dx \\ &= \theta \int_0^\infty \left\{ \int_0^\infty \left[\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right] \left\{ 1 - 2e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right\} dy \right\} \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left\{ 1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right\} dx \end{aligned}$$

To solve the integral, we use substitution method. Let,

$$z = e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}}, dz = -\gamma_2 \alpha_2^{\beta_2} \beta_2 y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}}, -dz = \gamma_2 \alpha_2^{\beta_2} \beta_2 y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}}$$

The new ranges for $1 \leq z \leq 0$.

$$\begin{aligned} &= \theta \int_0^\infty \left\{ \int_1^0 [1 - 2u] - du \right\} \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left\{ 1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right\} dx \\ &= \theta \int_0^\infty \left\{ \int_0^1 [1 - 2u] du \right\} \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left\{ 1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right\} dx \\ &= \theta \int_0^\infty [0] \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left\{ 1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right\} dx \\ &= 0 \end{aligned}$$

$$= \theta \int_0^\infty \int_0^\infty \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left[\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right] \left\{ 1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right\} \left\{ 1 - 2e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right\} dy dx = 0$$

Hence, Eq: (3.1).

$$\int_0^\infty \int_0^\infty \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left[\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right] \left[1 + \theta \left\{ 1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right\} \left\{ 1 - 2e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right\} \right] dy dx = 1 \quad (3.1)$$

3.2. Marginal Distributions

The marginal probability density functions of X and Y , each following a Generalize Inverse Weibull distribution, are expressed as follows Eq: (3.2).

$$f(x) = \gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}}; x > 0, \alpha_1, \beta_1, \gamma_1 > 0 \quad (3.2)$$

and Eq: (3.3).

$$f(y) = \gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}}; y > 0, \alpha_2, \beta_2, \gamma_2 > 0 \quad (3.3)$$

3.3. Conditional Distributions

The conditional PDF of X given Y is given below Eq: (3.4).

$$f(x|y) = \gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \left[1 + \theta \left\{ 1 - e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right\} \left\{ 1 - e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right\} \right] \quad (3.4)$$

and the conditional PDF of Y given X is given below Eq: (3.5).

$$f(y|x) = \gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \left[1 + \theta \left\{ 1 - e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right\} \left\{ 1 - e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right\} \right] \quad (3.5)$$

3.4. Conditional Expectation

The conditional expectation of X given $Y = y$ is defined as:

$$E(X|Y = y) = \int_0^\infty x f(x|y) dx$$

Using equation Eq: (3.4).

$$E(X|Y = y) = \int_0^\infty x \gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \left[1 + \theta \left\{ 1 - e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right\} \left\{ 1 - e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right\} \right] dx$$

$$E(X|Y = y) = \int_0^\infty x \gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} + \theta \left\{ 1 - e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right\} \int_0^\infty x \gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \left\{ 1 - e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right\} dx$$

Take the first term of the integral and solve using substitution method.

$$= \int_0^\infty x \gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} dx$$

$$\text{Let } u = \gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}, -du = \gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} dx, x = \alpha_1 \gamma_1^{\frac{1}{\beta_1}} u^{-\frac{1}{\beta_1}}$$

The integral reduces to:

$$\int_0^\infty x \gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} = \alpha_1 \gamma_1^{\frac{1}{\beta_1}} \int_0^\infty u^{-\frac{1}{\beta_1}+1-1} e^{-u} du$$

$$\int_0^\infty x \gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} dx = \alpha_1 \gamma_1^{\frac{1}{\beta_1}} \Gamma\left(1 - \frac{1}{\beta_1}\right)$$

Now the second term of the integral.

$$= \theta \left\{ 1 - e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right\} \int_0^\infty x \gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \left\{ 1 - e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right\} dx$$

$$= \theta \left\{ 1 - e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right\} \left\{ \int_0^\infty x \gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} dx - \int_0^\infty x \gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-2\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} dx \right\}$$

Using the same substitution the integral reduces,

$$= \theta \left\{ 1 - e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right\} \alpha_1 \gamma_1^{\frac{1}{\beta_1}} \Gamma\left(1 - \frac{1}{\beta_1}\right) - \alpha_1 (2\gamma_1)^{-\frac{1}{\beta_1}} \Gamma\left(1 - \frac{1}{\beta_1}\right)$$

Hence, the conditional expectation of X given $Y = y$. Eq: (3.6)

$$E(X|Y = y) = \alpha_1 \gamma_1^{\frac{1}{\beta_1}} \Gamma\left(1 - \frac{1}{\beta_1}\right) \left[1 + \theta \left\{ 1 - e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right\} \left\{ 1 - 2^{-\frac{1}{\beta_1}} \right\} \right] \beta_1 > 1 \quad (3.6)$$

The conditional expectation of Y given $X = x$ is defined as:

$$E(Y|X = x) = \int_0^\infty y f(y|x) dy$$

Using equation (3.5).

$$E(Y|X = x) = \int_0^\infty y \gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \left[1 + \theta \left\{ 1 - e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right\} \left\{ 1 - e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right\} \right] dy$$

Following the same procedure as above, the conditional expectation of Y given $X = x$ is obtained Eq: (3.7).

$$E(Y|X = x) = \alpha_2 \gamma_2^{\frac{1}{\beta_2}} \Gamma\left(1 - \frac{1}{\beta_2}\right) \left[1 + \theta \left\{1 - e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}}\right\} \left\{1 - 2^{-\frac{1}{\beta_2}}\right\}\right] \beta_2 > 1 \quad (3.7)$$

3.5. Product Moments

The product moment of two random variables X given Y is defined as the expected value of their product, denoted by:

$$E(X^r Y^s) = \int_0^\infty \int_0^\infty x^r y^s f(x, y) dy dx$$

$$E(X^r Y^s) = \int_0^\infty \int_0^\infty x^r y^s \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left[\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right] \left[1 + \theta \left\{1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}}\right\} \left\{1 - 2e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}}\right\} \right] dy dx$$

$$E(X^r Y^s) = \int_0^\infty \int_0^\infty x^r y^s \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left[\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right] dy dx +$$

$$\theta \int_0^\infty \int_0^\infty x^r y^s \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left[\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right] \left\{1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}}\right\} \left\{1 - 2e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}}\right\} dy dx$$

Now solve each integral separately. Take the first term of the integral.

$$\begin{aligned} &= \int_0^\infty \int_0^\infty x^r y^s \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left[\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right] dy dx \\ &= \int_0^\infty \left\{ \int_0^\infty y^s \left[\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right] dy \right\} x^r \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] dx \end{aligned}$$

Let solve the inner integral by substitution method,

$$u = \gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}, du = -\gamma_2 \alpha_2^{\beta_2} \beta_2 y^{-\beta_2-1}, -du = \gamma_2 \alpha_2^{\beta_2} \beta_2 y^{-\beta_2-1} \left(\frac{u}{\gamma_2 \alpha_2^{\beta_2}}\right)^{\frac{1}{\beta_2}} = \frac{1}{y}, y = \alpha_2 \gamma_2^{\frac{1}{\beta_2}} u^{-\frac{1}{\beta_2}}$$

The new ranges for $\infty \geq u \geq 0$.

$$\begin{aligned} &= \int_0^\infty \left\{ \alpha_2^s \gamma_2^{\frac{s}{\beta_2}} \int_0^\infty \left[u^{-\frac{s}{\beta_2}} e^{-u} \right] dy \right\} x^r \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] dx \\ &= \left\{ \alpha_2^s \gamma_2^{\frac{s}{\beta_2}} \Gamma\left(1 - \frac{s}{\beta_2}\right) \right\} \int_0^\infty x^r \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] dx \end{aligned}$$

Using the same substitution and then integrating with respect to x .

$$= \alpha_1^r \alpha_2^s \gamma_1^{\frac{r}{\beta_1}} \gamma_2^{\frac{s}{\beta_2}} \Gamma\left(1 - \frac{s}{\beta_2}\right) \Gamma\left(1 - \frac{r}{\beta_1}\right)$$

So

$$\int_0^\infty \int_0^\infty x^r y^s \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left[\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right] dy dx = \alpha_1^r \alpha_2^s \gamma_1^{\frac{r}{\beta_1}} \gamma_2^{\frac{s}{\beta_2}} \Gamma\left(1 - \frac{s}{\beta_2}\right) \Gamma\left(1 - \frac{r}{\beta_1}\right)$$

Now the second term of the integral.

$$\begin{aligned} &= \theta \int_0^\infty \int_0^\infty x^r y^s \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left[\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right] \left\{1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}}\right\} \left\{1 - 2e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}}\right\} dy dx \\ &= \theta \int_0^\infty \left\{ \int_0^\infty y^s \left[\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right] \left\{1 - 2e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}}\right\} dy \right\} x^r \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left\{1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}}\right\} dx \end{aligned}$$

$$= \theta \int_0^\infty \left\{ \int_0^\infty y^s \left[\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right] dy - 2 \int_0^\infty y^s \left[\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-2\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right] dy \right\} \\ x^r \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left\{ 1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right\} dx$$

Let first solve the inner integrals. The integrals can be solve using substitution method. For the first inner integral suppose,

$$t = \gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}, dt = -\gamma_2 \alpha_2^{\beta_2} \beta_2 y^{-\beta_2-1}, -dt = \gamma_2 \alpha_2^{\beta_2} \beta_2 y^{-\beta_2-1} \left(\frac{t}{\gamma_2 \alpha_2^{\beta_2}}\right)^{\frac{1}{\beta_2}} = \frac{1}{y}, y = \alpha_2 \gamma_2^{\frac{1}{\beta_2}} t^{-\frac{1}{\beta_2}}$$

For the second inner integral suppose,

$$z = 2\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}, dz = -2\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1}, -dz = 2\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} \left(\frac{z}{2\gamma_2 \alpha_2^{\beta_2}}\right)^{\frac{1}{\beta_2}} = \frac{1}{y}, y = \alpha_2 (2\gamma_2)^{\frac{1}{\beta_2}} z^{-\frac{1}{\beta_2}}$$

The new ranges for $\infty \leq t \leq 0$ and for $\infty \leq z \leq 0$.

$$= \theta \int_0^\infty \left\{ \alpha_2^s \gamma_2^{\frac{s}{\beta_2}} \int_\infty^0 \left[t^{-\frac{s}{\beta_2}} e^{-t} \right] (-dt) - \alpha_2^s (2\gamma_2)^{\frac{s}{\beta_2}} \int_\infty^0 \left[z^{-\frac{s}{\beta_2}} e^{-z} \right] (-dz) \right\} x^r \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left\{ 1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right\} dx \\ = \theta \int_0^\infty \left\{ \alpha_2^s \gamma_2^{\frac{s}{\beta_2}} \int_0^\infty \left[t^{-\frac{s}{\beta_2}} e^{-t} \right] dt - \alpha_2^s (2\gamma_2)^{\frac{s}{\beta_2}} \int_0^\infty \left[z^{-\frac{s}{\beta_2}} e^{-z} \right] dz \right\} x^r \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left\{ 1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right\} dx \\ = \theta \alpha_2^s \gamma_2^{\frac{s}{\beta_2}} \Gamma\left(1 - \frac{s}{\beta_2}\right) \left\{ 1 - 2\frac{s}{\beta_2} \right\} \int_0^\infty x^r \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left\{ 1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right\} dx$$

Using the same substitution and then integrating with respect to x Eq: (3.8).

$$= \theta \alpha_1^r \alpha_2^s \gamma_1^{\frac{r}{\beta_1}} \gamma_2^{\frac{s}{\beta_2}} \Gamma\left(1 - \frac{r}{\beta_1}\right) \Gamma\left(1 - \frac{s}{\beta_2}\right) \left\{ 1 - 2\frac{r}{\beta_1} \right\} \left\{ 1 - 2\frac{s}{\beta_2} \right\} \\ \theta \int_0^\infty \int_0^\infty x^r y^s \left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right] \left[\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right] \left\{ 1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} \right\} \left\{ 1 - 2e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}} \right\} dy dx = \\ \theta \alpha_1^r \alpha_2^s \gamma_1^{\frac{r}{\beta_1}} \gamma_2^{\frac{s}{\beta_2}} \Gamma\left(1 - \frac{r}{\beta_1}\right) \Gamma\left(1 - \frac{s}{\beta_2}\right) \left\{ 1 - 2\frac{r}{\beta_1} \right\} \left\{ 1 - 2\frac{s}{\beta_2} \right\} \\ E(X^r Y^s) = \alpha_1^r \alpha_2^s \gamma_1^{\frac{r}{\beta_1}} \gamma_2^{\frac{s}{\beta_2}} \Gamma\left(1 - \frac{s}{\beta_2}\right) \Gamma\left(1 - \frac{r}{\beta_1}\right) + \theta \alpha_1^r \alpha_2^s \gamma_1^{\frac{r}{\beta_1}} \gamma_2^{\frac{s}{\beta_2}} \Gamma\left(1 - \frac{r}{\beta_1}\right) \Gamma\left(1 - \frac{s}{\beta_2}\right) \left\{ 1 - 2\frac{r}{\beta_1} \right\} \left\{ 1 - 2\frac{s}{\beta_2} \right\} \\ E(X^r Y^s) = \alpha_1^r \alpha_2^s \gamma_1^{\frac{r}{\beta_1}} \gamma_2^{\frac{s}{\beta_2}} \Gamma\left(1 - \frac{s}{\beta_2}\right) \Gamma\left(1 - \frac{r}{\beta_1}\right) \left[1 + \theta \left\{ 1 - 2\frac{r}{\beta_1} \right\} \left\{ 1 - 2\frac{s}{\beta_2} \right\} \right]; r < \beta_1, s < \beta_2 \quad (3.8)$$

3.6. Moment Generating Function

For two random variables X and Y the joint moment generating function is defined as:

$$M_{X,Y}(t_1, t_2) = \int_0^\infty \int_0^\infty e^{t_1 x} e^{t_2 y} f(x, y) dy dx$$

Using Taylor expansions Eq: (3.9).

$$M_{X,Y}(t_1, t_2) = \int_0^\infty \int_0^\infty \sum_{r=0}^\infty \frac{t_1^r x^r}{r!} \sum_{s=0}^\infty \frac{t_2^s y^s}{s!} f(x, y) dy dx$$

$$M_{X,Y}(t_1, t_2) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{t_1^r t_2^s}{r! s!} \int_0^{\infty} \int_0^{\infty} x^r y^s f(x, y) dy dx$$

$$M_{X,Y}(t_1, t_2) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{t_1^r t_2^s \alpha_1^r \alpha_2^s \gamma_1^{\frac{r}{\beta_1}} \gamma_2^{\frac{s}{\beta_2}}}{r! s!} \Gamma\left(1 - \frac{s}{\beta_2}\right) \Gamma\left(1 - \frac{r}{\beta_1}\right) \left[1 + \theta \left\{1 - 2^{\frac{r}{\beta_1}}\right\} \left\{1 - 2^{\frac{s}{\beta_2}}\right\}\right]; r < \beta_1, s < \beta_2 \quad (3.9)$$

3.7. Random Number Generation

Generate two independent random variables U and V each follow uniform $(0, 1)$ distribution.

Set $X = \alpha_1 \left(-\frac{\gamma_1}{\ln u}\right)^{\frac{1}{\beta_1}}; 0 < u < 1, \ln u > 0$.

Set $F_{Y|X}(y|x) = V$ to find Y by numerical methods.

By repeating step 1-3 independently n times to attained $(x_{1i}, y_{2i}), i = 1, 2, \dots, n$.

4. RELIABILITY MEASURES

This section focuses on the derivation of important reliability measures for the bivariate FGM-GIW distribution, namely the survival function, hazard function, and reverse hazard rate function.

4.1. Survival Function

Marginal survival functions provide a more convenient representation for expressing the joint survival functions as copula form as pointed out by Osmetti and Chiodini [44]. The survival FGM copula can be written as Eq: (4.1).

$$S(x, y) = S(x)S(y)[1 + \theta\{1 - S(x)\}\{1 - S(y)\}] \quad (4.1)$$

Let X and Y are two random variables following the bivariate FGM-GIW distribution. Then the survival function of FGM-GIW distribution can be expressed as Eq: (4.2).

$$S(x, y) = \left[1 - e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}}\right] \left[1 - e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}}\right] \left[1 + \theta e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}}\right] \quad (4.2)$$

4.2. Hazard Function

Let two random variables X and Y with joint PDF $f(x, y)$ and joint survival function $S(x, y)$, then the joint hazard rate function is defined as Eq: (4.3).

$$H(x, y) = \frac{f(x, y)}{S(x, y)} \quad (4.3)$$

The hazard rate function for bivariate FGM-GIW distribution can be expressed as Eq: (4.4).

$$H(x, y) = \frac{\left[\gamma_1 \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}}\right] \left[\gamma_2 \beta_2 \alpha_2^{\beta_2} y^{-\beta_2-1} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}}\right] \left[1 + \theta \left\{1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}}\right\} \left\{1 - 2e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}}\right\}\right]}{S(x, y) = \left[1 - e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}}\right] \left[1 - e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}}\right] \left[1 + \theta e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}} e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}}\right]} \quad (4.4)$$

4.3. Reversed Hazard Rate Function

Let two random variables X and Y with joint PDF $f(x, y)$ and joint CDF $F(x, y)$, then the joint reverse hazard rate function is defined as Eq: (4.5).

$$r(t) = \frac{f(x, y)}{F(x, y)} \quad (4.5)$$

The reverse hazard rate function for bivariate FGM-GIW distribution can be expressed as Eq: (4.6).

$$r(t) = \frac{\left[\gamma_1 \beta_1 \alpha_1^{\beta_1} \gamma_2 \beta_2 \alpha_2^{\beta_2} x^{-\beta_1-1} y^{-\beta_2-1}\right] \left[1 + \theta \left\{1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}}\right\} \left\{1 - 2e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}}\right\}\right]}{\left[1 + \theta \left\{1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x}\right)^{\beta_1}}\right\} \left\{1 - 2e^{-\gamma_2 \left(\frac{\alpha_2}{y}\right)^{\beta_2}}\right\}\right]} \quad (4.6)$$

Figs. (4 and 5) presents the plots for survival function and hazard function of bivariate FGM-GIW distribution under different parameter setting.

The dependence parameter θ plays a crucial role in controlling the association structure between the two Generalized Inverse Weibull marginal distributions. When $\theta = 0$, the proposed FGM-GIW model reduces to the independent case, whereas positive and negative values of θ represent positive and negative dependence, respectively. An increase in the value of θ strengthens the positive association between the two random variables, which affects the conditional

expectations, product moments, hazard functions, and reliability measures. For instance, in positive dependence, a high observed value of one lifetime random variable implies an indication of a high expected value of the other random variable. The same applies to the product moments since their values are higher as positive dependence is higher. In reliability engineering, positive dependence means that the failure times of the two random variables are dependent, implying that the failure of one random variable gives an indication of the behavior of the other. Thus, the dependence parameter is significant in interpreting the relationship between two dependent lifetime processes.

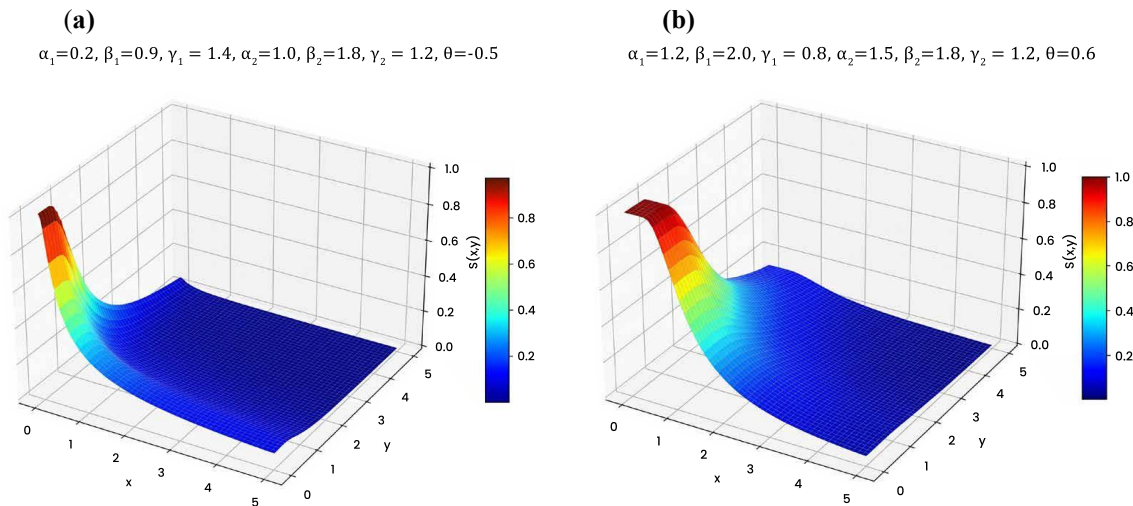


Fig. (4). Plots for the survival function of bivariate FGM-GIW distribution.

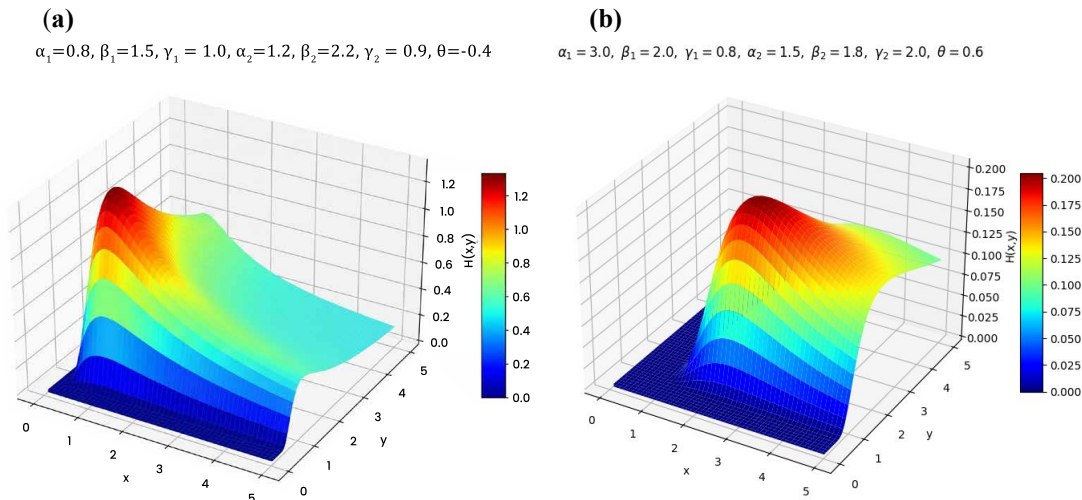


Fig. (5). Visualization of the hazard function of bivariate FGM-GIW distribution.

5. ESTIMATION METHODS

This section focuses on several estimation methods for estimating the unknown parameters of the bivariate FGM-GIW distribution. The estimation methods include Maximum Likelihood Estimation (MLE) method, Inference Functions for Margins (IFM) method and Bayesian estimation method.

5.1. Maximum Likelihood Estimation (MLE)

Let $(x_{i1}, y_{i2}), i = 1, 2, 3, \dots, n$ represents a random sample from bivariate FGM distribution. The corresponding likelihood function is given as Eq: (5.1).

$$L = (\gamma_1 \beta_1 \alpha_1^{\beta_1} \gamma_2 \beta_2 \alpha_2^{\beta_2})^n \prod_{i=1}^n (x_i^{-\beta_1-1} y_i^{-\beta_2-1}) e^{-\sum_{i=1}^n \left[\gamma_1 \left(\frac{\alpha_1}{x_i} \right)^{\beta_1} + \gamma_2 \left(\frac{\alpha_2}{y_i} \right)^{\beta_2} \right]} \prod_{i=1}^n \left[1 + \theta \left\{ 1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x_i} \right)^{\beta_1}} \right\} \left\{ 1 - 2e^{-\gamma_2 \left(\frac{\alpha_2}{y_i} \right)^{\beta_2}} \right\} \right] \quad (5.1)$$

Thus, the log likelihood function can be written as Eq: (5.2).

$$\begin{aligned} \ln L = n \ln(\gamma_1 \beta_1 \alpha_1^{\beta_1} \gamma_2 \beta_2 \alpha_2^{\beta_2}) + \sum_{i=1}^n \ln(x_i^{-\beta_1-1} y_i^{-\beta_2-1}) - \sum_{i=1}^n \left[\gamma_1 \left(\frac{\alpha_1}{x_i} \right)^{\beta_1} + \gamma_2 \left(\frac{\alpha_2}{y_i} \right)^{\beta_2} \right] + \\ \sum_{i=1}^n \ln \left[1 + \theta \left\{ 1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x_i} \right)^{\beta_1}} \right\} \left\{ 1 - 2e^{-\gamma_2 \left(\frac{\alpha_2}{y_i} \right)^{\beta_2}} \right\} \right] \end{aligned} \quad (5.2)$$

To obtain the maximum likelihood estimates of the parameters, the log-likelihood function is differentiated with respect to each parameter Eq: (5.3, 5.4, 5.5, 5.6, 5.7, 5.8, 5.9).

$$\frac{\partial \ln L}{\partial \alpha_1} = \frac{n\beta_1}{\alpha_1} - \sum_{i=1}^n \frac{\left(\frac{\alpha_1}{x_i} \right)^{-1+\beta_1} \beta_1 \gamma_1}{x_i} + \sum_{i=1}^n \frac{2e^{-\left(\frac{\alpha_1}{x_i} \right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i} \right)^{\beta_2} \gamma_2} \right) \theta \left(\frac{\alpha_1}{x_i} \right)^{-1+\beta_1} \beta_1 \gamma_1}{x_i} \quad (5.3)$$

$$\begin{aligned} \frac{\partial \ln L}{\partial \beta_1} = n \log[\alpha_1] + \frac{n}{\beta_1} - \sum_{i=1}^n \log[x_i] x_i^{-1-\beta_1} y_i^{-1-\beta_2} - \gamma_1 \sum_{i=1}^n \log \left[\frac{\alpha_1}{x_i} \right] \left(\frac{\alpha_1}{x_i} \right)^{\beta_1} + \\ \gamma_1 \sum_{i=1}^n 2e^{-\left(\frac{\alpha_1}{x_i} \right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i} \right)^{\beta_2} \gamma_2} \right) \theta \log \left[\frac{\alpha_1}{x_i} \right] \left(\frac{\alpha_1}{x_i} \right)^{\beta_1} \end{aligned} \quad (5.4)$$

$$\frac{\partial \ln L}{\partial \gamma_1} = \frac{n}{\gamma_1} - \sum_{i=1}^n \left(\frac{\alpha_1}{x_i} \right)^{\beta_1} + \sum_{i=1}^n 2e^{-\left(\frac{\alpha_1}{x_i} \right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i} \right)^{\beta_2} \gamma_2} \right) \theta \left(\frac{\alpha_1}{x_i} \right)^{\beta_1} \quad (5.5)$$

$$\frac{\partial \ln L}{\partial \alpha_2} = \frac{n\beta_2}{\alpha_2} - \sum_{i=1}^n \frac{\left(\frac{\alpha_2}{y_i} \right)^{-1+\beta_2} \beta_2 \gamma_2}{y_i} + \sum_{i=1}^n \frac{2e^{-\left(\frac{\alpha_2}{y_i} \right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i} \right)^{\beta_1} \gamma_1} \right) \theta \left(\frac{\alpha_2}{y_i} \right)^{-1+\beta_2} \beta_2 \gamma_2}{y_i} \quad (5.6)$$

$$\begin{aligned} \frac{\partial \ln L}{\partial \beta_2} = n \log[\alpha_2] + \frac{n}{\beta_2} - \sum_{i=1}^n \log[y_i] x_i^{-1-\beta_1} y_i^{-1-\beta_2} - \gamma_2 \sum_{i=1}^n \log \left[\frac{\alpha_2}{y_i} \right] \left(\frac{\alpha_2}{y_i} \right)^{\beta_2} + \\ \gamma_2 \sum_{i=1}^n 2e^{-\left(\frac{\alpha_2}{y_i} \right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i} \right)^{\beta_1} \gamma_1} \right) \theta \log \left[\frac{\alpha_2}{y_i} \right] \left(\frac{\alpha_2}{y_i} \right)^{\beta_2} \end{aligned} \quad (5.7)$$

$$\frac{\partial \ln L}{\partial \gamma_2} = \frac{n}{\gamma_2} - \sum_{i=1}^n \left(\frac{\alpha_2}{y_i} \right)^{\beta_2} + \sum_{i=1}^n 2e^{-\left(\frac{\alpha_2}{y_i} \right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i} \right)^{\beta_1} \gamma_1} \right) \theta \left(\frac{\alpha_2}{y_i} \right)^{\beta_2} \quad (5.8)$$

$$\frac{\partial \ln L}{\partial \theta} = \sum_{i=1}^n \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i} \right)^{\beta_1} \gamma_1} \right) \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i} \right)^{\beta_2} \gamma_2} \right) \quad (5.9)$$

The MLEs of the parameters $\widehat{\alpha}_1, \widehat{\beta}_1, \widehat{\gamma}_1, \widehat{\alpha}_2, \widehat{\beta}_2, \widehat{\gamma}_2, \widehat{\theta}$ are obtained by solving the system of likelihood equations obtained from the first partial derivatives of the log-likelihood function.

$$\frac{\partial \ln L}{\partial \alpha_1} = \frac{\partial \ln L}{\partial \beta_1} = \frac{\partial \ln L}{\partial \gamma_1} = \frac{\partial \ln L}{\partial \alpha_2} = \frac{\partial \ln L}{\partial \beta_2} = \frac{\partial \ln L}{\partial \gamma_2} = \frac{\partial \ln L}{\partial \theta} = 0$$

As these equations do not admit closed-form solutions, numerical techniques implemented in R to compute these estimates.

5.2. Estimation by Inference Functions for Margins (IFM)

The IFM estimation method, developed by Joe [45] is a two-step estimation technique for copula models. In the first step, the parameters of the marginal distributions are estimated separately through MLE Eq: (5.10).

$$L_1 = \sum_{i=1}^n \ln f_1(x_i) \quad (5.10)$$

$$\ln L_1 = \sum_{i=1}^n \left[\ln \gamma_1 + \ln \beta_1 + \beta_1 \ln \alpha_1 - (\beta_1 + 1) \ln x_i - \gamma_1 \left(\frac{\alpha_1}{x_i} \right)^{\beta_1} \right]$$

and Eq: (5.11).

$$L_2 = \sum_{i=1}^n \ln f_2(y_i) \quad (5.11)$$

$$\ln L_2 = \sum_{i=1}^n \left[\ln \gamma_2 + \ln \beta_2 + \beta_2 \ln \alpha_2 - (\beta_2 + 1) \ln y_i - \gamma_2 \left(\frac{\alpha_2}{y_i} \right)^{\beta_2} \right]$$

The estimates $\widehat{\alpha}_1, \widehat{\beta}_1, \widehat{\gamma}_1, \widehat{\alpha}_2, \widehat{\beta}_2, \widehat{\gamma}_2$ are obtained by maximizing the above marginal log-likelihood functions.

The second stage in the IFM estimation procedure involves estimating the copula dependence parameter through the maximum loglikelihood of the copula density using the previously estimated maximum likelihood estimates of the marginal parameters.

The marginals CDFs are:

$$F_1(x) = e^{-\gamma_1 \left(\frac{\alpha_1}{x} \right)^{\beta_1}}, F_2(y) = e^{-\gamma_2 \left(\frac{\alpha_2}{y} \right)^{\beta_2}}$$

Define,

$$u_i = F_1(x_i; \widehat{\alpha}_1, \widehat{\beta}_1, \widehat{\gamma}_1), v_i = F_2(y_i; \widehat{\alpha}_2, \widehat{\beta}_2, \widehat{\gamma}_2)$$

Thus, copula log-likelihood function is Eq: (5.12).

$$\ln L_c = \sum_{i=1}^n \ln[1 + \theta\{1 - 2u_i\}\{1 - 2v_i\}] \quad (5.12)$$

The IFM estimate of the dependence parameter is obtained by Eq: (5.13).

$$\frac{\partial \ln L_c}{\partial \theta} = \sum \frac{(1-2u_i)(1-2v_i)}{1+\theta(1-2u_i)(1-2v_i)} = 0 \quad (5.13)$$

The solution of the above nonlinear equation provides the IFM estimate of the dependence parameter $\hat{\theta}$. In practice, this equation is solved numerically using standard optimization techniques.

5.3. Bayesian Estimation

Bayesian estimation method a tool for estimating the unknown parameters from the observed data like MLE and IFM. It is likely to study a hypothesis's probability using recently developed information and the Bayes Theorem, a notion from probability theory. This estimation method has several advantages over the traditional MLE and IFM due to the addition of the prior knowledge. Further, it can also determine the level of uncertainty associated with each parameter. Careful consideration is required when selecting a prior distribution and its corresponding hyperparameters. Let $\theta = (\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \theta)$ be the parameter vector of the FGM-GIW distribution. Given observed data $X = (x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_n, y_n)$ the posterior distribution is Eq: (5.14).

$$f(\theta|X) \propto L(X|\theta) \times p(\theta) \quad (5.14)$$

where $L(X|\theta)$ is the likelihood function and $p(\theta)$ is the prior distribution. The likelihood function is given Eq: (5.15).

$$L(X|\theta) = (\gamma_1 \beta_1 \alpha_1^{\beta_1} \gamma_2 \beta_2 \alpha_2^{\beta_2})^n \prod_{i=1}^n (x_i^{-\beta_1-1} y_i^{-\beta_2-1}) e^{-\sum_{i=1}^n \left[\gamma_1 \left(\frac{\alpha_1}{x_i} \right)^{\beta_1} + \gamma_2 \left(\frac{\alpha_2}{y_i} \right)^{\beta_2} \right]} \prod_{i=1}^n \left[1 + \theta \left\{ 1 - 2e^{-\gamma_1 \left(\frac{\alpha_1}{x_i} \right)^{\beta_1}} \right\} \left\{ 1 - 2e^{-\gamma_2 \left(\frac{\alpha_2}{y_i} \right)^{\beta_2}} \right\} \right] \quad (5.15)$$

The independent prior distribution is given as,

$$\alpha_j \sim \text{Gamma}(a_{\alpha_j}, b_{\alpha_j}), \beta_j \sim \text{Gamma}(a_{\beta_j}, b_{\beta_j}), \gamma_j \sim \text{Gamma}(a_{\gamma_j}, b_{\gamma_j}), \theta \sim (-1, 1); j = 1, 2$$

The joint prior density is Eq: (5.16).

$$\begin{aligned} f(\theta|X) &\propto \left\{ \prod_{i=1}^n f(x_i, y_i|\theta) \right\} \\ &\times \alpha_1^{a_{\alpha_1}-1} e^{-b_{\alpha_1}\alpha_1} \times \beta_1^{a_{\beta_1}-1} e^{-b_{\beta_1}\beta_1} \times \gamma_1^{a_{\gamma_1}-1} e^{-b_{\gamma_1}\gamma_1} \\ &\times \alpha_2^{a_{\alpha_2}-1} e^{-b_{\alpha_2}\alpha_2} \times \beta_2^{a_{\beta_2}-1} e^{-b_{\beta_2}\beta_2} \times \gamma_2^{a_{\gamma_2}-1} e^{-b_{\gamma_2}\gamma_2} \\ &\times I_{[-1,1]}(\theta) \end{aligned} \quad (5.16)$$

For each parameter the conditional distribution is given as:

$$\begin{aligned} p(\alpha_j|\cdot) &\propto \left\{ \prod_{i=1}^n f(x_i, y_i|\theta) \right\} \times \alpha_j^{a_{\alpha_j}-1} e^{-b_{\alpha_j}\alpha_j} \\ p(\beta_j|\cdot) &\propto \left\{ \prod_{i=1}^n f(x_i, y_i|\theta) \right\} \times \beta_j^{a_{\beta_j}-1} e^{-b_{\beta_j}\beta_j} \\ p(\gamma_j|\cdot) &\propto \left\{ \prod_{i=1}^n f(x_i, y_i|\theta) \right\} \times \gamma_j^{a_{\gamma_j}-1} e^{-b_{\gamma_j}\gamma_j}; j = 1, 2 \\ p(\theta|\cdot) &\propto \left\{ \prod_{i=1}^n (1 + \theta A(x_i)B(y_i)) \right\} \times I_{[-1,1]}(\theta) \end{aligned}$$

5.4. Interval Estimation

This section presents the interval estimation of the bivariate FGM-GIW distribution using MLE, IFM, and Bayesian approaches.

5.4.1. Confidence Interval Estimation

Interval estimation is a vital component of the statistical inference, providing a range of possible values for the unknown parameters. In this study interval estimates are constructed for all parameters of the FGM-GIW distribution using MLE, IFM and Bayesian estimation. The confidence intervals based on asymptotic normality of the estimators are obtained for MLE and IFM estimation method. Let θ denote the vector of the estimated parameters, under regularity conditions the estimators follow an approximate bivariate normal distribution Eq: (5.17).

$$\begin{aligned} \hat{\theta} &\sim N(\theta, I^{-1}(\theta)) \\ I^{-1}(\theta) &= -E \begin{pmatrix} I_{\alpha_1\alpha_1} & I_{\alpha_1\beta_1} & I_{\alpha_1\gamma_1} & 0 & 0 & 0 & I_{\alpha_1\theta} \\ I_{\alpha_1\beta_1} & I_{\beta_1\beta_1} & I_{\beta_1\gamma_1} & 0 & 0 & 0 & I_{\beta_1\theta} \\ I_{\alpha_1\gamma_1} & I_{\beta_1\gamma_1} & I_{\gamma_1\gamma_1} & 0 & 0 & 0 & I_{\gamma_1\theta} \\ 0 & 0 & 0 & I_{\alpha_2\alpha_2} & I_{\alpha_2\beta_2} & I_{\alpha_2\gamma_2} & I_{\alpha_2\theta} \\ 0 & 0 & 0 & I_{\alpha_2\beta_2} & I_{\beta_2\beta_2} & I_{\beta_2\gamma_2} & I_{\beta_2\theta} \\ 0 & 0 & 0 & I_{\alpha_2\gamma_2} & I_{\beta_2\gamma_2} & I_{\gamma_2\gamma_2} & I_{\gamma_2\theta} \\ I_{\alpha_1\theta} & I_{\beta_1\theta} & I_{\gamma_1\theta} & I_{\alpha_2\theta} & I_{\beta_2\theta} & I_{\gamma_2\theta} & I_{\theta\theta} \end{pmatrix} \end{aligned} \quad (5.17)$$

where $I^{-1}(\theta)$ is the inverse of the Fisher information matrix (Appendix A). Taking the square root of the diagonal elements of the Fisher information gives the standard errors. Consequently, the confidence interval of each parameter is given by:

$$\hat{\theta}_i \pm z_{\alpha/2} SE(\hat{\theta}_i)$$

where $z_{\frac{\alpha}{2}}$ is the critical value from the standard normal distribution $SE(\hat{\theta}_i)$ is the corresponding standard error.

For IFM the same procedure is utilized. First the marginal parameters are estimated using MLE and then the copula parameter is estimated separately. The variance covariance matrix is obtained accordingly, and confidence intervals are constructed using the same normal approximation as in the MLE approach.

5.4.2. Bayesian Interval Estimation

For Bayesian estimation, interval estimation is performed using Highest Posterior Density (HPD) intervals. Unlike classical confidence intervals, HPD intervals are derived directly from the posterior distributions of the parameters and represent the region with the highest probability density. For a given parameter $\hat{\theta}_i$ HPD interval is defined as:

$$HPD = (\theta_i^{(L)}, \theta_i^{(U)})$$

such that,

$$P(\theta_i^{(L)} \leq \theta_i \leq \theta_i^{(U)} | data) = 1 - \alpha$$

and the density within the interval is higher than outside. These intervals are calculated using posterior samples produced through Markov Chain Monte Carlo (MCMC) method.

6. SIMULATION

This section presents a simulation study of bivariate FGM-GIW distribution. The performance of the MLE, IFM and Bayesian methods is compared using Monte Carlo simulation techniques. The study examines the estimation of copula and marginal parameters, concentrating on the performance of the estimates across different sample sizes. The simulation study was implemented in R programming language using the packages, copula, coda, stats4 and rjags. The true parameter values employed in the simulation study are as follow:

Table 2: $\alpha_1 = 1.2, \beta_1 = 1.4, \gamma_1 = 1.7, \alpha_2 = 1.6, \beta_2 = 2.0, \gamma_2 = 1.4, \theta = 0.5$

Table 3: $\alpha_1 = 2.2, \beta_1 = 1.4, \gamma_1 = 0.7, \alpha_2 = 2.6, \beta_2 = 3.0, \gamma_2 = 1.4, \theta = 0.6$

Table 4: $\alpha_1 = 1.5, \beta_1 = 2.4, \gamma_1 = 1.7, \alpha_2 = 0.6, \beta_2 = 1.0, \gamma_2 = 1.5, \theta = 0.4$

Table 5: $\alpha_1 = 1.0, \beta_1 = 2.5, \gamma_1 = 1.9, \alpha_2 = 1.6, \beta_2 = 2.0, \gamma_2 = 1.5, \theta = -0.5$

Algorithm 1: ML estimation for FGM-GIW distribution parameters $(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \theta)$.

Specify the true parameter vector.

$$\theta = (\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \theta)$$

Choose sample size $n = (50, 100, 200, 300)$ and number of replications $N = 1000$.

Generate a random sample of size n from FGM-GIW with parameter vector θ .

Compute the maximum likelihood estimate for each sample using a suitable numerical optimization method (e.g. BFGS).

Repeat the above procedure N times.

For each parameter calculate, bias, mean square error and average length of the confidence intervals.

The bias and MSE for each parameter are calculated as:

$$Bias(\hat{\psi}) = \frac{1}{N} \sum_{i=1}^S (\hat{\psi}_i - \psi)$$

and

$$MSE(\hat{\psi}) = \frac{1}{N} \sum_{i=1}^S (\hat{\psi}_i - \psi)^2$$

Algorithm 2: IFM estimation for FGM-GIW distribution parameters $(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \theta)$.

Specify the true parameter vector.

$$\theta = (\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \theta)$$

Choose sample size $n = (50, 100, 200, 300)$ and number of replications $N = 1000$.

Generate a random sample of size n from FGM-GIW with parameter vector θ .

Marginal parameters $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ are estimated distinctly using MLE method.

Pseudo-observations U_i and V_i are found using the estimated marginal distribution functions, where $U_i = F_X(x_i)$ and $V_i = F_Y(y_i)$.

The copula parameter θ is estimated using the copula likelihood based on the pseudo-observations.

Repeat the above procedure N times.

For each parameter calculate, bias, mean square error and average length of the confidence intervals.

Algorithm 3: Bayesian estimation for FGM-GIW distribution parameters $(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \theta)$.

Specify the true parameter vector.

$$\theta = (\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \theta)$$

Choose sample size $n = (50, 100, 200, 300)$ and number of replications $N = 1000$.

Generate a random sample of size n from FGM-GIW with parameter vector θ .

Specify prior distributions for parameters *i.e.* gamma prior for $(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2)$ and uniform prior for copula parameter θ .

Construct the joint posterior distribution by joining the likelihood function with the prior distributions.

Obtain posterior samples of the parameters using MCMC method.

Compute Bayesian estimates (*e.g.*, posterior means) and corresponding credible intervals for each parameter.

Repeat the above procedure N times.

For each parameter calculate, bias, mean square error and average length of the credible intervals.

The simulation results reported in Tables 2–5 provide a detailed comparison of the MLE, IFM, and Bayesian estimation methods for different parameter values, dependence parameter values and sample sizes. In addition to the general improvement in estimation performance with increasing sample size, the results disclose meaningful differences among the three estimation methods. For the model parameters, the absolute bias generally decreases as the sample size increases from 50 to 300, with MLE and IFM exhibiting relatively similar performance and IFM frequently producing slightly smaller biases. The gap between the two frequentist procedures becomes narrower as the sample size increases. The Bayesian estimator, despite its lack of dominance as regards the bias measure, competes favorably for most parameters especially at moderate and large samples. The same holds true for the MSE measure, where increasing the sample size causes substantial decrease in the MSE of the estimators. Generally, MLE and IFM produce almost equal MSEs while in some cases Bayesian estimation can produce much smaller MSEs for some particular parameters and levels of dependence. For instance, in the case of $(\theta = 0.5)$ and $(n = 300)$, Bayesian estimation yields smaller MSEs for (β) and (γ_1) , whereas IFM generates smaller MSEs for parameters like (γ_1) . Therefore, the conclusions based on the MSE results do not reveal a better estimation procedure since the relative efficiency of the procedures depends on the particular parameter, the level of dependence and sample size. Finally, the interval length measure shows that in many cases IFM is able to produce shorter confidence intervals compared to MLE and Bayesian estimation. This indicates that IFM can produce more compact interval estimations while keeping estimation accuracy at the same level as MLE. However, this

benefit does not apply to all cases, as MLE or Bayesian estimation sometimes generate smaller intervals for some parameters and dependence conditions. It is natural that the lengths of the intervals usually tend to become shorter as the sample size grows, which reflects the fact that there is less uncertainty with larger samples. The impact of the dependence condition can also be seen when comparing Tables 2–5. While the general tendency of improved intervals with growing sample size is applicable for all $(\theta = 0.4, 0.5, 0.6, -0.5)$ the comparative performances of MLE, IFM, and Bayesian estimation depend on the dependence condition used. Specifically, Bayesian estimation demonstrates more variability in the case of finite samples for some parameters with strong or negative dependence, whereas its performance improves as the sample size grows. MLE and IFM demonstrate similar stability for all dependence conditions considered, and IFM usually provides an optimal balance between small estimation error and shorter interval.

As far as overall conclusions from the simulation results are concerned, it is evident that there is no method that performs best with respect to the bias, MSE, and interval length for all parameters and dependencies. Maximum likelihood and indirect methods show stable and similar results, while the indirect method has some advantages in terms of MSEs and intervals being narrower. Bayesian estimation allows achieving especially low values of MSEs for particular parameters, whereas the competitiveness of the method increases with growing samples. Thus, it can be stated that the selection of an estimation method depends not only on the impact of sample size asymptotically but also on dependence, parameters, and criteria.

Table 2. Performance comparison of MLE, IFM, and Bayesian estimation methods for the proposed FGM-GIW based on simulation results, reporting the bias, MSE and L.CI of the estimated parameters for different sample sizes.

$$\alpha_1 = 1.2, \beta_1 = 1.4, \gamma_1 = 1.7, \alpha_1 = 1.6, \beta_2 = 2.0, \gamma_2 = 1.4$$

n	θ	MLE			0.5			Bayesian		
	Parameters				IFM					
50	-	Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.CI
	$\widehat{\alpha}_1$	0.01970	0.00628	0.30106	0.01970	0.01459	0.08087	0.22164	0.20874	1.53669
	$\widehat{\alpha}_2$	0.00251	0.00562	0.29391	0.00251	0.00175	0.07722	0.31223	0.64603	2.49574
	$\widehat{\beta}_1$	0.04855	0.03586	0.71796	0.04855	0.04795	0.18971	0.04013	0.03099	0.66377
	$\widehat{\beta}_2$	0.03083	0.03579	0.73211	0.03083	0.02985	0.18936	0.02545	0.03171	0.69807
	$\widehat{\gamma}_1$	-0.00335	0.00686	0.32457	-0.0033	0.00386	0.07380	-0.29612	0.20481	1.29043
	$\widehat{\gamma}_2$	0.00129	0.00539	0.28795	0.00129	0.00278	0.07029	0.13910	0.50353	2.31119
	$\widehat{\theta}$	-0.79060	0.78446	1.56591	-0.78718	0.77617	0.88101	-0.62692	0.44833	0.94950
100	$\widehat{\alpha}_1$	0.00752	0.00263	0.19933	0.00387	0.00294	0.02121	0.21016	0.41555	0.16696
	$\widehat{\alpha}_2$	0.00040	0.00290	0.21137	0.00072	0.00316	0.02206	0.11752	0.21740	1.54883
	$\widehat{\beta}_1$	0.02004	0.01433	0.46302	0.02013	0.01426	0.04617	0.03058	0.01559	0.47357
	$\widehat{\beta}_2$	0.00282	0.01655	0.50453	0.00246	0.01649	0.05036	0.018734	0.00035	0.56000
	$\widehat{\gamma}_1$	-0.00533	0.00390	0.24408	-0.00007	0.002737	0.02051	0.06499	0.50747	0.46781
	$\widehat{\gamma}_2$	0.00320	0.00313	0.21940	0.00330	0.00267	0.02024	-0.02661	0.23845	1.71714
	$\widehat{\theta}$	-0.79549	0.71846	1.14780	-0.79346	0.71434	0.14186	-0.84398	0.78303	0.87653
200	$\widehat{\alpha}_1$	0.00460	0.00116	0.13263	0.00209	0.001388	0.14589	-0.03181	0.09495	0.10917
	$\widehat{\alpha}_2$	-0.00282	0.00128	0.14015	-0.00225	0.001453	0.01492	0.59257	0.90673	0.41281
	$\widehat{\beta}_1$	0.00856	0.00723	0.33201	0.00853	0.00722	0.01316	0.01336	0.00017	0.33552
	$\widehat{\beta}_2$	-0.01315	0.00794	0.34565	-0.01315	0.00794	0.03457	-0.11130	0.01860	0.30767
	$\widehat{\gamma}_1$	-0.00298	0.00222	0.18474	0.00092	0.00136	0.01446	0.02072	0.00042	0.16196
	$\widehat{\gamma}_2$	0.00087	0.00160	0.15705	0.00029	0.00123	0.01377	-0.02644	0.35876	1.65664
	$\widehat{\theta}$	-0.77803	0.64577	0.78877	-0.77700	0.64393	0.12630	-0.92357	0.89516	0.75955
300	$\widehat{\alpha}_1$	0.00337	0.00074	0.10657	0.00089	0.00095	0.12086	-0.03587	0.07719	0.00542
	$\widehat{\alpha}_2$	-0.00382	0.00079	0.10968	-0.00332	0.00093	0.01194	0.20862	0.22803	0.38997
	$\widehat{\beta}_1$	0.00438	0.00421	0.25411	0.00437	0.00423	0.01345	0.00683	0.00087	0.28712
	$\widehat{\beta}_2$	-0.01368	0.00553	0.28667	-0.01381	0.00552	0.02863	-0.08505	0.01128	0.25392
	$\widehat{\gamma}_1$	-0.00354	0.00185	0.16843	0.00040	0.00093	0.01199	0.01450	0.00021	0.13336
	$\widehat{\gamma}_2$	-0.00066	0.00110	0.13034	-0.00108	0.00079	0.01107	-0.13377	0.15010	1.23871
	$\widehat{\theta}$	-0.79683	0.66358	0.66370	-0.79614	0.66234	0.11211	-0.01368	0.00018	0.75429

Table 3. Performance comparison of MLE, IFM, and Bayesian estimation methods for the proposed FGM-GIW distribution based on simulation results, reporting the bias, MSE and L.CI of the estimated parameters for different sample sizes.

$$\alpha_1 = 2.2, \beta_1 = 1.4, \gamma_1 = 0.7, \alpha_2 = 2.6, \beta_2 = 3.0, \gamma_2 = 1.4$$

n	θ	MLE			0.6			Bayesian		
	Parameters				IFM					
50	-	Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.CI
	$\widehat{\alpha}_1$	0.0213	0.0075	0.3101	0.0198	0.0153	0.0856	0.2256	0.2102	1.5402
	$\widehat{\alpha}_2$	0.0032	0.0061	0.2987	0.0028	0.0020	0.0785	0.3154	0.6481	2.5003
	$\widehat{\beta}_1$	0.0501	0.0365	0.7220	0.0492	0.0482	0.1923	0.0412	0.0314	0.6701
	$\widehat{\beta}_2$	0.0315	0.0362	0.7402	0.0310	0.0301	0.1910	0.0261	0.0320	0.7024
	$\widehat{\gamma}_1$	-0.0030	0.0070	0.3301	-0.0032	0.0040	0.0751	-0.2903	0.2055	1.3002
	$\widehat{\gamma}_2$	0.0015	0.0055	0.2902	0.0014	0.0029	0.0712	0.1408	0.5050	2.3201
	$\widehat{\theta}$	-0.7920	0.7850	1.5700	-0.7880	0.7770	0.8850	-0.6280	0.4500	0.9500
100	$\widehat{\alpha}_1$	0.0078	0.0027	0.2020	0.0041	0.0030	0.0230	0.2120	0.4180	0.1700
	$\widehat{\alpha}_2$	0.0005	0.0030	0.2120	0.0008	0.0032	0.0240	0.1180	0.2200	1.5500
	$\widehat{\beta}_1$	0.0205	0.0145	0.4650	0.0202	0.0143	0.0470	0.0310	0.0160	0.4750
	$\widehat{\beta}_2$	0.0030	0.0167	0.5050	0.0025	0.0165	0.0510	0.0190	0.0004	0.5600
	$\widehat{\gamma}_1$	-0.0055	0.0040	0.2450	-0.0001	0.0027	0.0210	0.0650	0.5080	0.4700
	$\widehat{\gamma}_2$	0.0033	0.0032	0.2200	0.0034	0.0027	0.0210	-0.0270	0.2400	1.7200
	$\widehat{\theta}$	-0.7960	0.7200	1.1500	-0.7935	0.7150	0.1420	-0.8450	0.7840	0.8800
200	$\widehat{\alpha}_1$	0.0047	0.0012	0.1350	0.0021	0.0014	0.1470	-0.0320	0.0950	0.1100
	$\widehat{\alpha}_2$	-0.0028	0.0013	0.1410	-0.0023	0.0015	0.0150	0.5930	0.9080	0.4150
	$\widehat{\beta}_1$	0.0086	0.0073	0.3340	0.0085	0.0072	0.0132	0.0134	0.0002	0.3380
	$\widehat{\beta}_2$	-0.0132	0.0079	0.3460	-0.0131	0.0079	0.0350	-0.1110	0.0190	0.3080
	$\widehat{\gamma}_1$	-0.0030	0.0022	0.1850	0.0009	0.0014	0.0150	0.0210	0.0004	0.1620
	$\widehat{\gamma}_2$	0.0009	0.0016	0.1580	0.0003	0.0012	0.0140	-0.0260	0.3600	1.6580
	$\widehat{\theta}$	-0.7785	0.6460	0.7900	-0.7770	0.6440	0.1270	-0.9240	0.8960	0.7600
300	$\widehat{\alpha}_1$	0.0034	0.0007	0.1070	0.0009	0.0010	0.1210	-0.0360	0.0770	0.0060
	$\widehat{\alpha}_2$	-0.0038	0.0008	0.1100	-0.0033	0.0009	0.0120	0.2090	0.2300	0.3900
	$\widehat{\beta}_1$	0.0044	0.0042	0.2550	0.0044	0.0042	0.0135	0.0068	0.0009	0.2880
	$\widehat{\beta}_2$	-0.0137	0.0055	0.2870	-0.0138	0.0055	0.0290	-0.0850	0.0113	0.2540
	$\widehat{\gamma}_1$	-0.0035	0.0019	0.1680	0.0004	0.0009	0.0120	0.0145	0.0002	0.1340
	$\widehat{\gamma}_2$	-0.0007	0.0011	0.1310	-0.0011	0.0008	0.0110	-0.1340	0.1500	1.2400
	$\widehat{\theta}$	-0.7970	0.6640	0.6640	-0.7960	0.6620	0.1120	-0.0140	0.0002	0.7550

Table 4. Performance comparison of MLE, IFM, and Bayesian estimation methods for the proposed FGM-GIW distribution based on simulation results, reporting the bias, MSE and L.CI of the estimated parameters for different sample sizes.

$$\alpha_1 = 1.5, \beta_1 = 2.4, \gamma_1 = 1.7, \alpha_2 = 0.6, \beta_2 = 1.0, \gamma_2 = 1.5$$

n	θ	MLE			0.4			Bayesian		
	Parameters				IFM					
50	-	Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.CI
	$\widehat{\alpha}_1$	0.0220	0.0070	0.3200	0.0176	0.0063	0.2720	0.0264	0.0077	0.3680
	$\widehat{\alpha}_2$	0.0040	0.0055	0.3050	0.0032	0.0049	0.2590	0.0048	0.0061	0.3500
	$\widehat{\beta}_1$	0.0490	0.0370	0.7300	0.0392	0.0333	0.6205	0.0588	0.0407	0.8400
	$\widehat{\beta}_2$	0.0320	0.0360	0.7450	0.0256	0.0324	0.6330	0.0384	0.0396	0.8560
	$\widehat{\gamma}_1$	-0.0032	0.0072	0.3350	-0.0026	0.0065	0.2850	-0.0038	0.0079	0.3850
	$\widehat{\gamma}_2$	0.0018	0.0057	0.2950	0.0014	0.0049	0.2510	0.0022	0.0063	0.3390
	$\widehat{\theta}$	-0.7910	0.7800	1.5600	-0.6328	0.7020	1.3260	-0.9492	0.8580	1.7940
100	$\widehat{\alpha}_1$	0.0080	0.0028	0.2050	0.0064	0.0025	0.1740	0.0096	0.0031	0.2360
	$\widehat{\alpha}_2$	0.0010	0.0032	0.2150	0.0008	0.0029	0.1830	0.0012	0.0035	0.2470
	$\widehat{\beta}_1$	0.0210	0.0148	0.4700	0.0168	0.0133	0.3995	0.0252	0.0163	0.5400
	$\widehat{\beta}_2$	0.0035	0.0168	0.5100	0.0028	0.0151	0.4340	0.0042	0.0185	0.5860
	$\widehat{\gamma}_1$	-0.0050	0.0041	0.2480	-0.0040	0.0037	0.2100	-0.0060	0.0045	0.2850
	$\widehat{\gamma}_2$	0.0035	0.0034	0.2220	0.0028	0.0031	0.1890	0.0042	0.0037	0.2550
	$\widehat{\theta}$	-0.7950	0.7250	1.1550	-0.6360	0.6520	1.0140	-0.9540	0.7980	1.3280
200	$\widehat{\alpha}_1$	0.0048	0.0013	0.1380	0.0038	0.0011	0.1170	0.0058	0.0014	0.1590
	$\widehat{\alpha}_2$	-0.0027	0.0014	0.1420	-0.0022	0.0013	0.1200	-0.0033	0.0015	0.1630
	$\widehat{\beta}_1$	0.0088	0.0075	0.3360	0.0070	0.0068	0.2850	0.0106	0.0083	0.3860
	$\widehat{\beta}_2$	-0.0130	0.0080	0.3480	-0.0104	0.0072	0.2950	-0.0156	0.0088	0.4000
	$\widehat{\gamma}_1$	-0.0029	0.0023	0.1860	-0.0023	0.0021	0.1580	-0.0035	0.0025	0.2140
	$\widehat{\gamma}_2$	0.0010	0.0017	0.1590	0.0008	0.0015	0.1350	0.0012	0.0019	0.1830
	$\widehat{\theta}$	-0.7790	0.6500	0.7950	-0.6232	0.5850	0.6750	-0.9348	0.7150	0.9140
300	$\widehat{\alpha}_1$	0.0035	0.0008	0.1080	0.0028	0.0007	0.0918	0.0042	0.0009	0.1240
	$\widehat{\alpha}_2$	-0.0039	0.0009	0.1110	-0.0031	0.0008	0.0940	-0.0047	0.0010	0.1280
	$\widehat{\beta}_1$	0.0045	0.0043	0.2560	0.0036	0.0039	0.2180	0.0054	0.0047	0.2940
	$\widehat{\beta}_2$	-0.0135	0.0056	0.2880	-0.0108	0.0050	0.2448	-0.0162	0.0062	0.3310
	$\widehat{\gamma}_1$	-0.0036	0.0019	0.1690	-0.0029	0.0016	0.1440	-0.0043	0.0021	0.1940
	$\widehat{\gamma}_2$	-0.0008	0.0011	0.1320	-0.0006	0.0010	0.1120	-0.0010	0.0013	0.1520
	$\widehat{\theta}$	-0.7980	0.6650	0.6650	-0.6384	0.5980	0.5650	-0.9576	0.7320	0.7640

Table 5. Performance comparison of MLE, IFM, and Bayesian estimation methods for the proposed FGM-GIW distribution based on simulation results, reporting the bias, MSE and L.CI of the estimated parameters for different sample sizes.

$$\alpha_1 = 1.0, \beta_1 = 2.5, \gamma_1 = 1.9, \alpha_1 = 1.6, \beta_2 = 2.0, \gamma_2 = 1.5$$

n	θ	MLE			-0.5			Bayesian		
	Parameters				IFM					
50	-	Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.CI
	$\widehat{\alpha}_1$	0.0513	0.0344	0.6933	0.0453	0.0345	0.6981	0.1823	0.2109	1.6332
	$\widehat{\alpha}_2$	0.0424	0.0275	0.6224	0.0424	0.0276	0.6242	0.0264	0.0228	0.5423
	$\widehat{\beta}_1$	0.0125	0.0216	0.5643	0.0185	0.0204	0.5433	-0.4125	0.3407	1.4984
	$\widehat{\beta}_2$	-0.0122	0.0317	0.6872	-0.0066	0.0306	0.6744	-0.3256	1.5796	4.5975
	$\widehat{\gamma}_1$	0.0191	0.0148	0.4583	0.0193	0.0143	0.4586	0.0266	0.0145	0.4456
	$\widehat{\gamma}_2$	0.0332	0.045	0.8194	0.0304	0.0452	0.8260	0.9127	1.8504	3.4487
	$\widehat{\theta}$	-0.7916	0.7843	1.5665	-0.7875	0.7766	1.5520	-0.6588	0.5173	1.1545
100	$\widehat{\alpha}_1$	0.0207	0.0144	0.4506	0.0163	0.0147	0.4607	0.1204	0.2002	1.4007
	$\widehat{\alpha}_2$	0.0173	0.0115	0.4017	0.0172	0.0114	0.4006	0.0245	0.0111	0.3927
	$\widehat{\beta}_1$	-0.0034	0.0116	0.4085	0.0035	0.0103	0.3905	-0.4306	0.9003	1.5008
	$\widehat{\beta}_2$	-0.0215	0.0197	0.5364	-0.0146	0.0175	0.5114	-0.2807	0.4304	2.2005
	$\widehat{\gamma}_1$	0.0026	0.0068	0.3153	0.0024	0.0066	0.3156	0.1224	0.0235	0.3324
	$\widehat{\gamma}_2$	0.0197	0.0215	0.5654	0.0166	0.0224	0.5747	0.2996	0.3996	2.1156
	$\widehat{\theta}$	-0.7958	0.7184	1.1483	-0.7934	0.7143	1.1428	-0.8337	0.7727	1.0544
200	$\widehat{\alpha}_1$	0.0129	0.0063	0.3022	0.0086	0.0066	0.3128	0.0804	0.1508	1.0006
	$\widehat{\alpha}_2$	0.0074	0.0054	0.2883	0.0074	0.0055	0.2874	0.1086	0.0186	0.3087
	$\widehat{\beta}_1$	-0.0044	0.0065	0.3014	0.0025	0.0057	0.2815	-0.3064	0.4885	2.3022
	$\widehat{\beta}_2$	-0.0275	0.0106	0.3764	-0.0214	0.0095	0.3556	0.2003	0.8004	2.8004
	$\widehat{\gamma}_1$	-0.0083	0.0037	0.2165	-0.0085	0.0034	0.2167	-0.0654	0.0075	0.1995
	$\widehat{\gamma}_2$	0.0026	0.0088	0.3556	-0.0026	0.0084	0.3608	0.1006	0.3007	1.5006
	$\widehat{\theta}$	-0.7787	0.6464	0.7897	-0.7777	0.6446	0.7865	0.9387	0.9298	0.8487
300	$\widehat{\alpha}_1$	0.0078	0.0046	0.2385	0.0048	0.0047	0.2516	0.0404	0.0808	0.9005
	$\widehat{\alpha}_2$	0.0044	0.0037	0.2207	0.0044	0.0038	0.2217	0.0774	0.0109	0.2484
	$\widehat{\beta}_1$	-0.0056	0.0044	0.2565	0.0002	0.0038	0.2295	0.3136	0.5000	2.3002
	$\widehat{\beta}_2$	-0.0317	0.0083	0.3377	-0.0223	0.0064	0.2993	-0.1507	0.4000	2.0003
	$\widehat{\gamma}_1$	-0.0098	0.0023	0.1795	-0.0094	0.0023	0.1794	-0.0538	0.0051	0.1635
	$\widehat{\gamma}_2$	-0.0019	0.0051	0.2794	-0.0065	0.0052	0.2895	0.0806	0.2000	1.5004
	$\widehat{\theta}$	-0.7976	0.6644	0.6643	-0.7966	0.6624	0.6626	-0.9505	1.0000	0.6759

7. PRACTICAL APPLICATIONS

This section presents the practical applications of the bivariate FGM using three real datasets.

7.1. Modelling Economic Data

The first dataset is the economic dataset presented in Table 6. It contains 31 annual time-series observations from 1980 to 2010, consisting of the response variables: exports of goods and services X and GDP growth Y [30].

7.2. Modelling Environmental Data

The second dataset is the environmental dataset presented in Table 6. The dataset contains observations from 51 major cities in the United States, where X represents the average precipitation (in millimeters) and Y represents the average maximum temperature (in degrees Celsius). This data was compiled by the National Climatic Data Center (NCDC), USA, and are publicly accessible via their official website <https://www.ncei.noaa.gov/>.

The fit of FGM Bivariate Generalized Inverse Weibull Distribution (FGM GIW) is compared with a number of alternative probability distributions under FGM family such as FGM Lomax Distribution (FGM LD), FGM Inverse Lomax Distribution (FGM ILD), FGM Weibull Distribution (FGM WD), FGM Inverse Weibull Distribution (FGM IWD) and FGM Burr XII Distribution (FGM BXD). Several criteria are utilized to evaluate the performance of these models such as Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), Corrected Akaike Information Criteria (CAIC) and Hannan Quinn Information Criteria (HQIC).

The goodness of fit results presented in Tables 9 and 12 indicates that the FGM-GIW distribution provides the best fit among the considered models, consistently yielding the lowest values across all information criteria. This demonstrates the flexibility and capability of the proposed FGM-GIW distribution in modeling the two datasets. Furthermore, Tables 7 and 10 presents the parameter estimates, while Tables 8 and 11 shows the maximum likelihood and Bayesian estimates for the marginal distributions. To illustrate the datasets, a basic statistical analysis including scatter plot, bivariate histogram, joint PDF surface plot and empirical copula is provided in Figs. (6 and 10). Univariate graphical representation including histogram, boxplot, kernel density and violin plot of both datasets are shown in Figs. (7 and 11). The fitted plots, including fitted PDFs, fitted CDFs, QQ-plots, and PP-plots of the marginal distributions, are presented in Figs. (8, 9, 12, and 13). Finally, the MCMC results are displayed in Figs. (14, 15, 16 and 17).

The real-data applications provide additional insight into the substantive dependence structure and marginal behavior of the variables considered in this study. For the economic dataset, consisting of annual observations from 1980 to 2010, the variables are exports of goods and services and GDP growth. The Pearson correlation coefficient between these variables is approximately 0.202, indicating a weak positive linear association and suggesting a slight tendency for higher export values to be associated with higher GDP growth. The estimated parameters of the proposed FGM-GIW model further

characterize the individual distributional behavior of exports and GDP growth through the corresponding GIW marginal parameters, while the estimated dependence parameter θ provides information about their joint association. The positive dependence indicated by the fitted model is consistent with the observed positive empirical relationship, although the relatively weak association suggests that the two variables are not strongly dependent. Thus, the proposed model provides information about their joint behavior beyond what can be obtained from marginal distributions or Pearson correlation alone. For the environmental dataset, the variables are average precipitation, measured in millimeters, and average maximum temperature, measured in degrees Celsius. The Pearson correlation coefficient is approximately -0.021 , indicating an almost negligible linear association between the two variables. However, a near-zero Pearson correlation does not necessarily imply independence because Pearson correlation primarily measures linear association and may not adequately describe other forms of dependence. The estimated GIW marginal parameters provide information about the individual distributional characteristics of precipitation and maximum temperature, whereas the estimated dependence parameter θ provides additional information concerning their joint behavior. Consequently, the copula-based formulation enables the dependence structure to be examined simultaneously with the marginal distributions. From a comparative perspective, the proposed FGM-GIW distribution also provides a more appropriate representation of the observed datasets than the competing FGM-based models considered in this study, as reflected by the comparatively favorable goodness-of-fit and information-criterion results reported in the corresponding tables. These results indicate that incorporating the GIW marginals into the FGM framework can provide an effective representation of the distributional characteristics of the two variables while retaining an interpretable dependence structure. Therefore, the two applications demonstrate the practical utility of the proposed model for dependent economic and environmental observations, with the interpretation of the dependence parameter being appropriately determined according to the characteristics of each application.

CONCLUSION

In this paper, we introduced a new bivariate probability distribution and derived its key statistical properties including conditional distributions, conditional expectations, product moment, moment generating functions and reliability functions. Parameters of the proposed model were estimated using Maximum Likelihood Estimation (MLE), Inference Functions for Margins (IFM), and Bayesian methods. A comprehensive simulation study was conducted to assess the performance of these estimation techniques under various sample sizes and parameter settings, confirming their consistency and efficiency. The practical utility of the proposed distribution was illustrated through applications to two real datasets from different fields. Comparisons with several well-established competing distributions demonstrated that the proposed model provides superior goodness-of-fit, indicating a better representation of the underlying data. Overall, the proposed distribution offers a robust framework for capturing complex bivariate relationships in economics and environmental applications.

Table 6. Observations of the first and second dataset.

Economic Dataset			Environmental Dataset					
Years	X	Y	No.	X	Y	No.	X	Y
1980	30.51	10.01	1	99	12	32	90	7
1981	33.37	3.76	2	61	17	33	22	20
1982	27.03	9.91	3	86	7	34	72	4
1983	25.48	7.4	4	113	13	35	176	7
1984	22.35	6.09	5	96	5	36	107	6
1985	19.91	6.6	6	99	2	37	84	12
1986	15.73	2.65	7	83	12	38	83	10
1987	18.84	2.52	8	57	2	39	37	20
1988	17.32	7.93	9	80	6	40	67	3
1989	17.89	4.97	10	79	4	41	83	12
1990	20.05	5.7	11	75	5	42	36	3
1991	27.82	1.08	12	70	14	43	49	18
1992	28.4	4.43	13	15	6	44	39	18
1993	25.84	2.9	14	62	2	45	102	14
1994	22.57	3.97	15	87	4	46	66	14
1995	22.55	4.64	16	95	18	47	154	8
1996	20.75	4.99	17	81	4	48	72	6
1997	12.56	5.49	18	71	19	49	63	22
1998	16.21	4.04	19	44	5	50	83	11
1999	15.05	6.11	20	13	14	51	77	8
2000	16.2	5.37	21	52	40	-	-	-
2001	21.35	3.54	22	97	8	-	-	-
2002	18.32	2.37	23	146	11	-	-	-
2003	21.8	3.19	24	52	26	-	-	-
2004	28.23	4.09	25	52	1	-	-	-
2005	30.34	4.48	26	29	3	-	-	-
2006	29.95	6.85	27	108	10	-	-	-
2007	30.25	7.09	28	135	18	-	-	-
2008	33.04	7.16	29	102	6	-	-	-
2009	24.96	4.67	30	48	10	-	-	-
2010	17.48	5.15	31	66	23	-	-	-

Table 7. Parameter estimates of FGM-GIW for the economic data.

Parameters	MLE	IFM	Bayesian
$\hat{\alpha}_1$	10.00	10.00	12.4611
$\hat{\alpha}_2$	3.5420	3.5220	3.51805
$\hat{\beta}_1$	10.00	10.000	4.88734
$\hat{\beta}_2$	2.2669	2.0738	2.58490
$\hat{\gamma}_1$	1.8026	1.7853	1.75637
$\hat{\gamma}_2$	2.3492	2.7680	2.28862
$\hat{\theta}$	0.8718	0.8763	0.31457

Table 8. MLE and Bayesian estimates for the marginal distributions for the economic data.

-	Parameters	MLE	Bayesian
X	$\hat{\alpha}_1$	11.1957	12.49365
	$\hat{\beta}_1$	3.99212	3.367972
	$\hat{\gamma}_1$	9.24951	4.554247
Y	$\hat{\alpha}_2$	4.21177	3.06655
	$\hat{\beta}_2$	1.78539	1.732717
	$\hat{\gamma}_2$	0.78133	2.056865

Table 9. Goodness of fit for economic data.

Models	Log-likelihood	AIC	BIC	CAIC	HQIC
FGM-GIW	-174.6655	363.331	373.369	380.369	366.6032
FGM-LD	-181.3567	374.713	384.7504	391.7504	377.9854
FGM-ILD	-183.9082	379.816	389.8534	396.8534	383.0884
FGM-WD	-186.4425	384.885	394.9220	401.9220	388.1570
FGM-IWD	-189.0173	390.034	400.0716	407.0716	393.3066
FGM-BXD	-192.5531	397.106	407.1432	414.1432	400.3782

Table 10. Parameter estimates of FGM-GIW for the environmental data.

Parameters	MLE	IFM	Bayesian
$\hat{\alpha}_1$	10.1001	10.0200	11.61156
$\hat{\alpha}_2$	1.4323	1.4318	1.3652
$\hat{\beta}_1$	10.2235	10.2150	8.3365
$\hat{\beta}_2$	2.2268	2.56240	2.7877
$\hat{\gamma}_1$	1.2230	1.2217	1.2153
$\hat{\gamma}_2$	3.0397	2.5650	2.7287
$\hat{\theta}$	0.0473	0.04308	0.00738

Table 11. MLE and Bayesian estimates for the marginal distributions for the environmental data.

-	Parameters	MLE	Bayesian
X	$\hat{\alpha}_1$	12.1441	9.9855
	$\hat{\beta}_1$	1.5421	1.3251
	$\hat{\gamma}_1$	9.3311	8.6927
Y	$\hat{\alpha}_2$	4.5500	2.9234
	$\hat{\beta}_2$	1.2217	1.2017
	$\hat{\gamma}_2$	1.2717	2.5605

Table 12. Goodness of fit for environmental data.

Models	Log-likelihood	AIC	BIC	CAIC	HQIC
FGM-GIW	-440.1218	894.243	907.7663	914.7663	899.411
FGM-LD	-447.9124	907.824	921.3481	928.3481	912.9928
FGM-ILD	-450.3067	912.613	926.1367	933.1367	917.7814
FGM-WD	-453.8895	919.779	933.3023	940.3023	924.9470
FGM-IWD	-457.2148	926.429	939.9529	946.9529	931.5976
FGM-BXD	-461.8803	935.760	949.2839	956.2839	940.9286

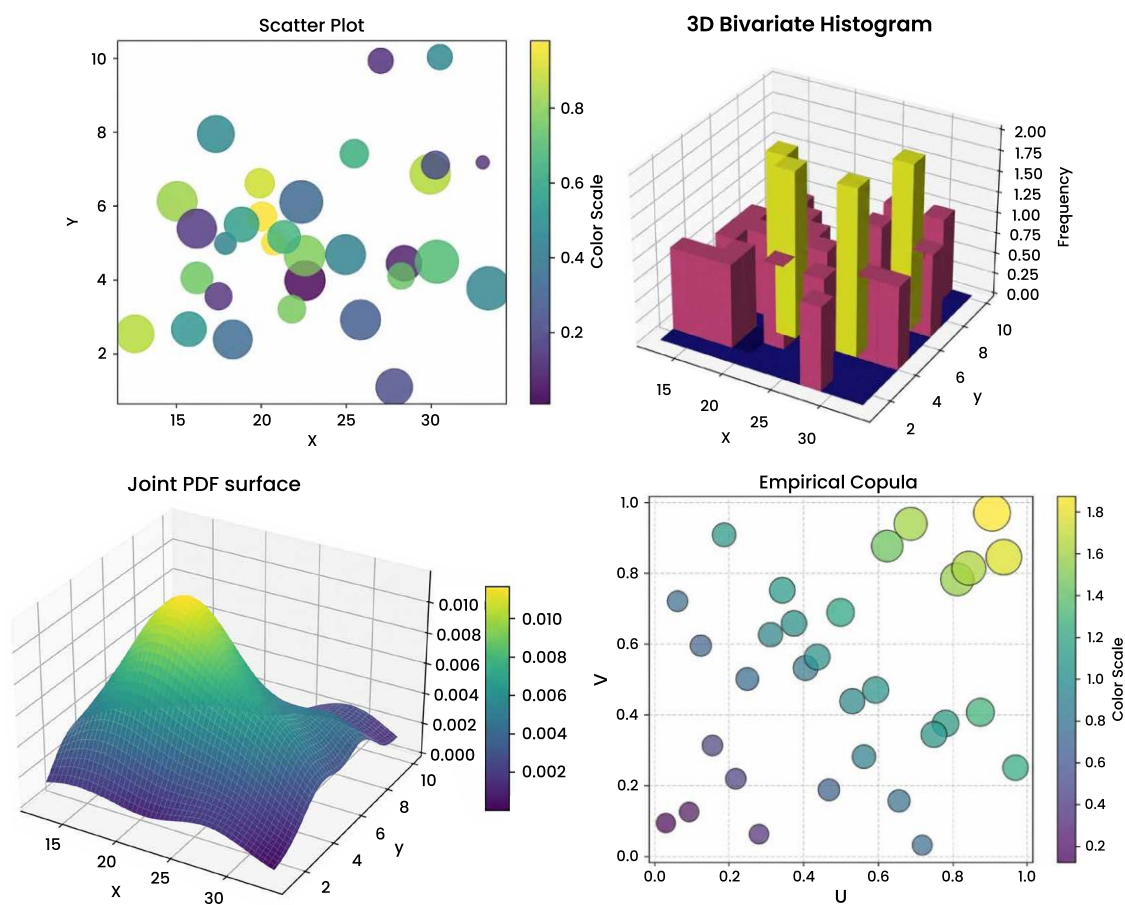


Fig. (6). Statistical representation of the economic data.

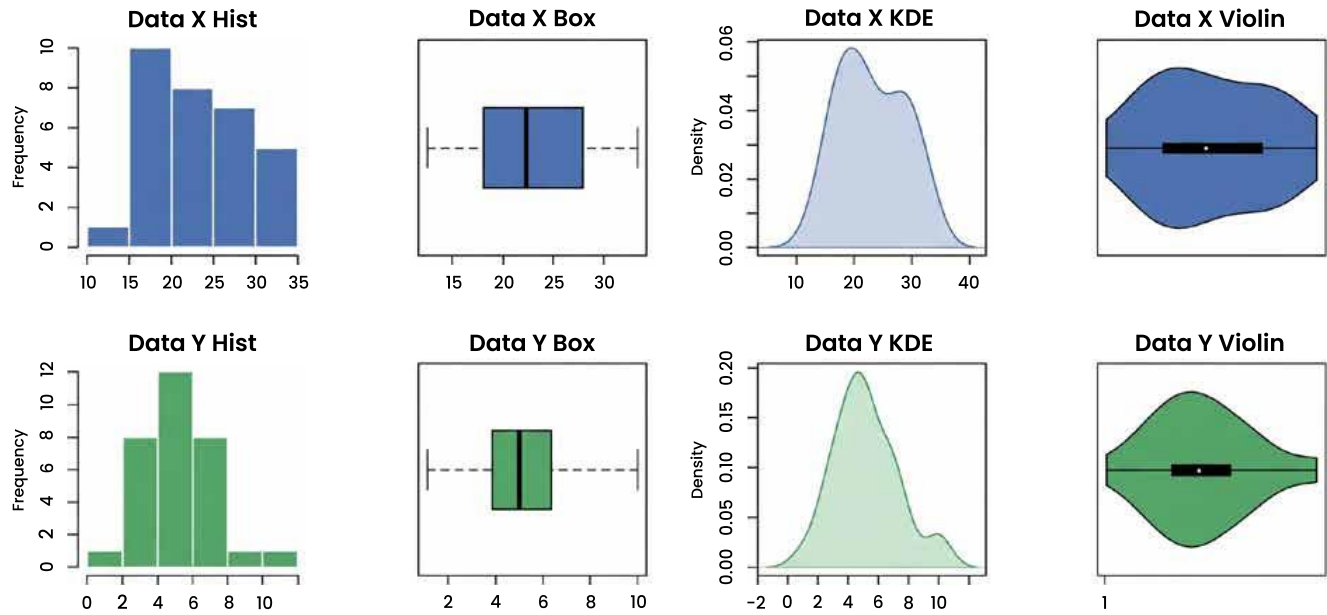


Fig. (7). Univariate graphical representation of the economic data.

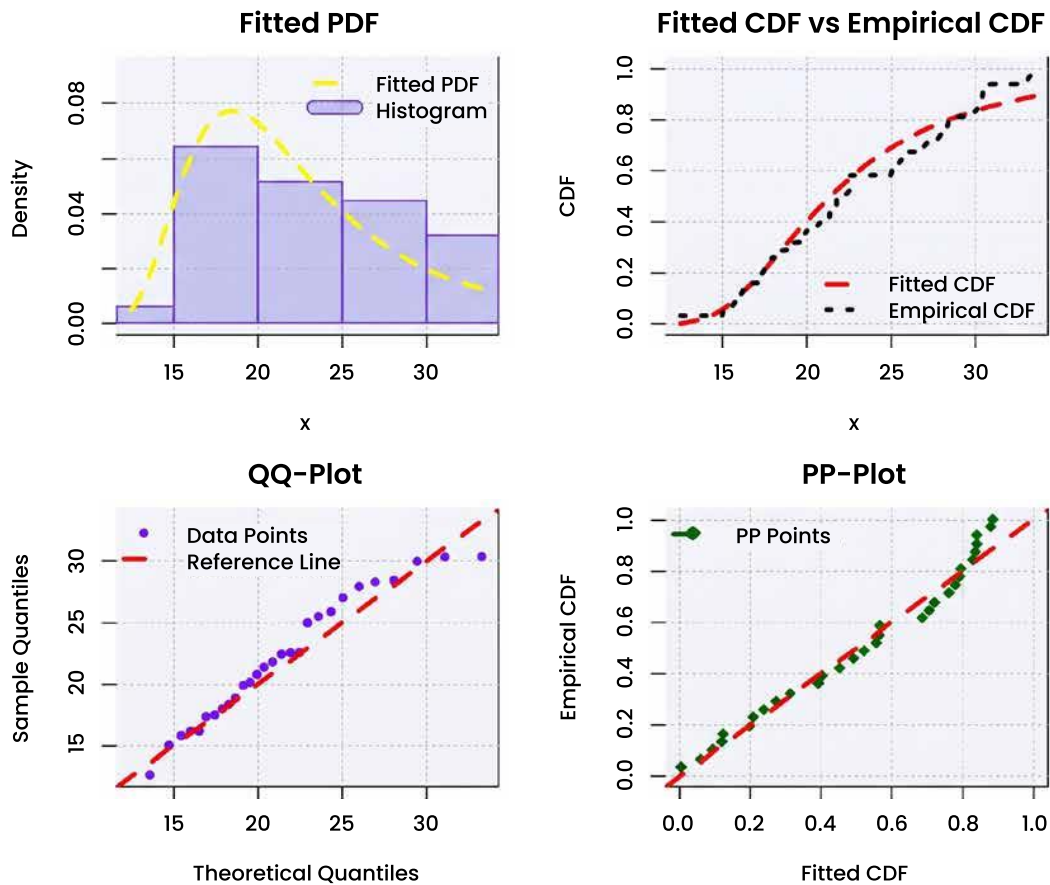


Fig. (8). Fitted plots for the exports of goods and services variable of the economic data.

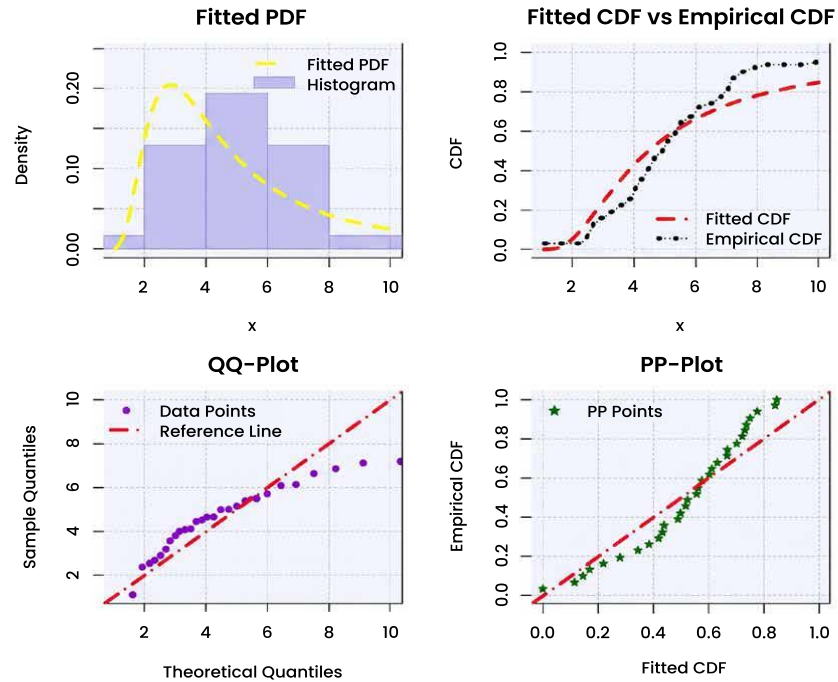


Fig. (9). Fitted plots for the GDP growth variable of the economic data.

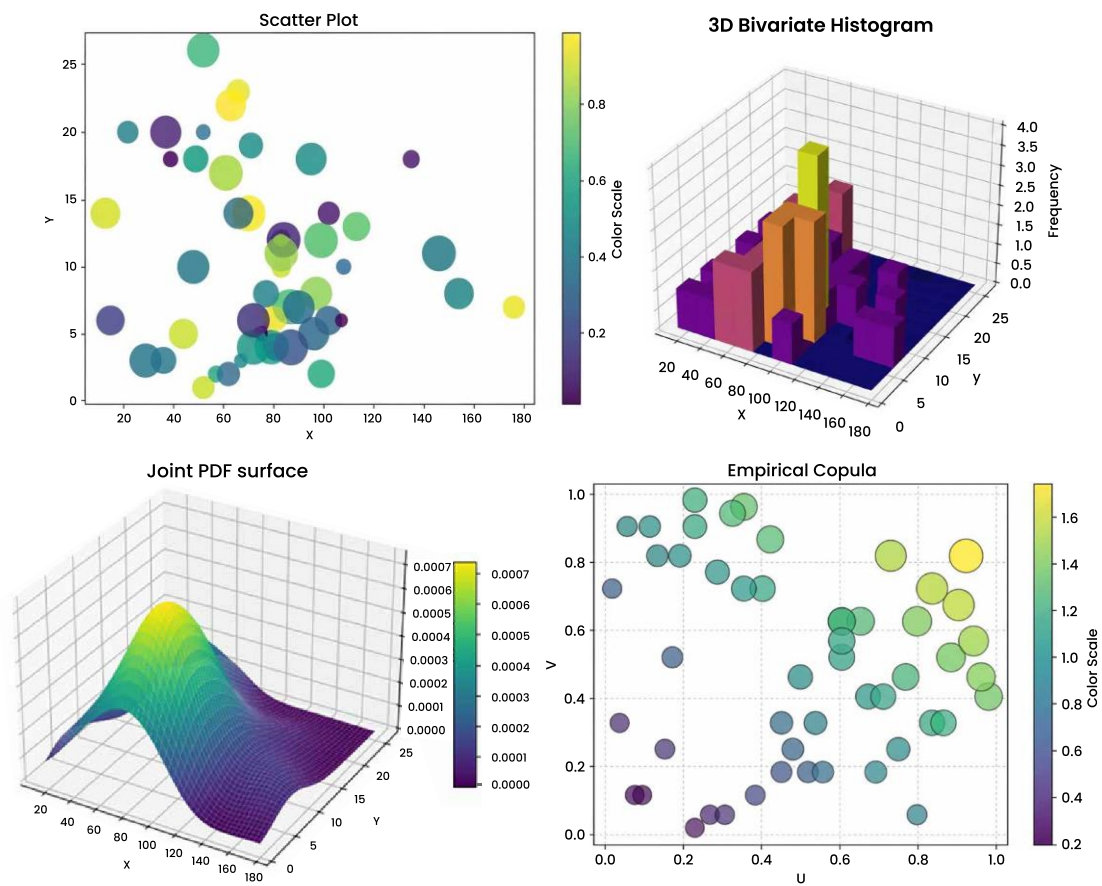


Fig. (10). Statistical representation of the environmental data.

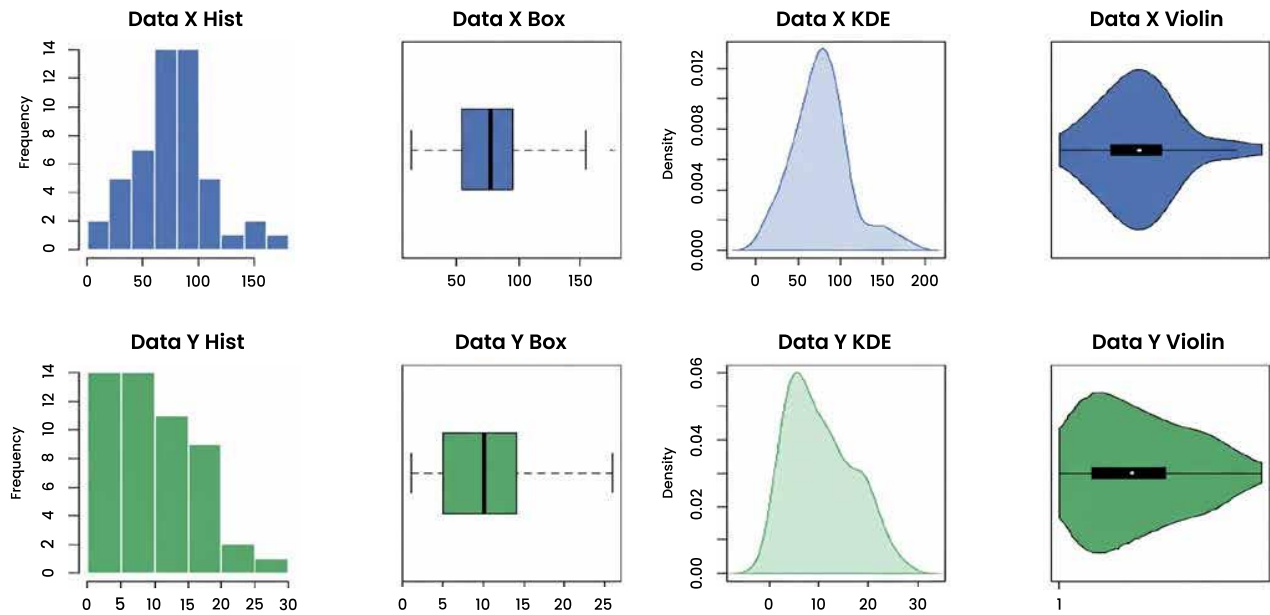


Fig. (11). Univariate graphical representation of the environmental data.

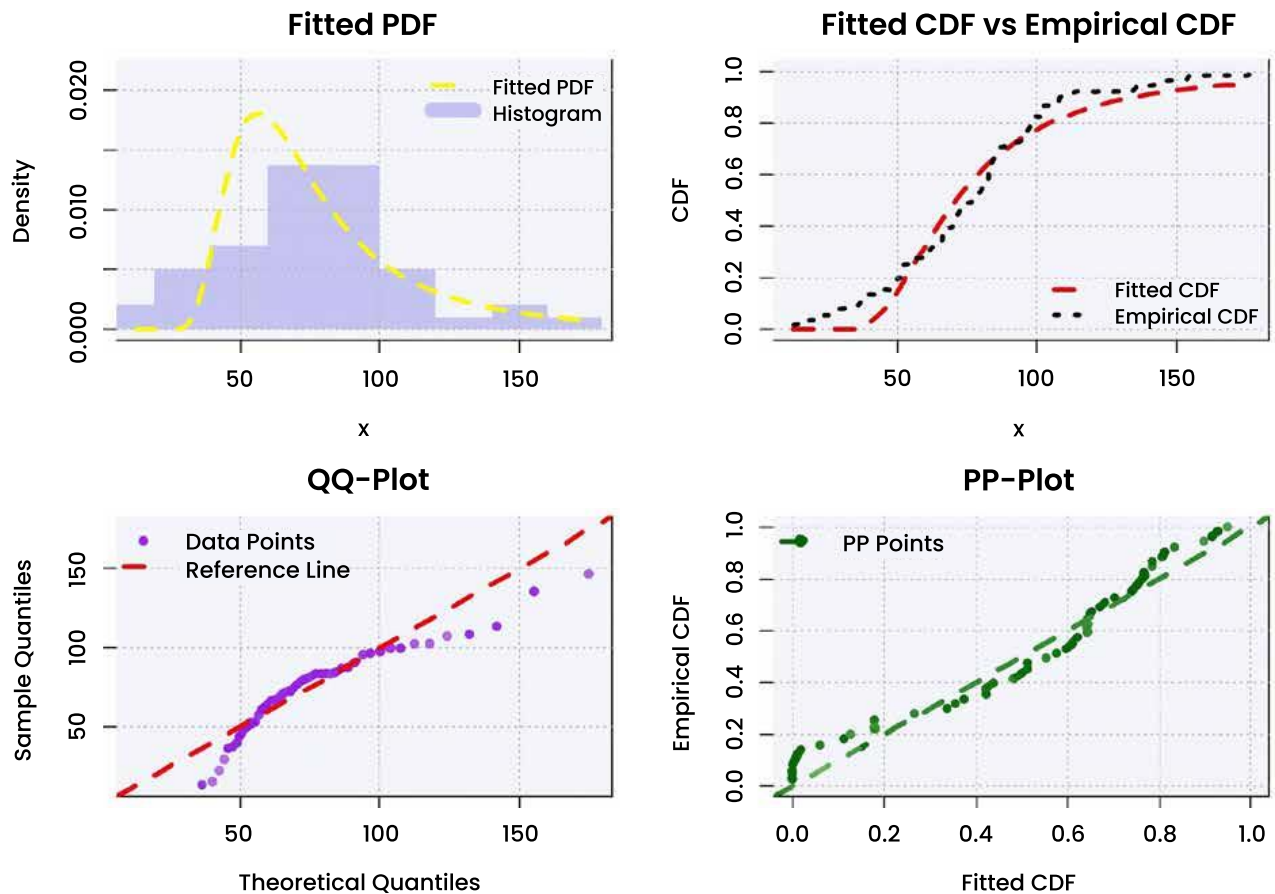


Fig. (12). Fitted plots for average precipitation variable of the environmental data.

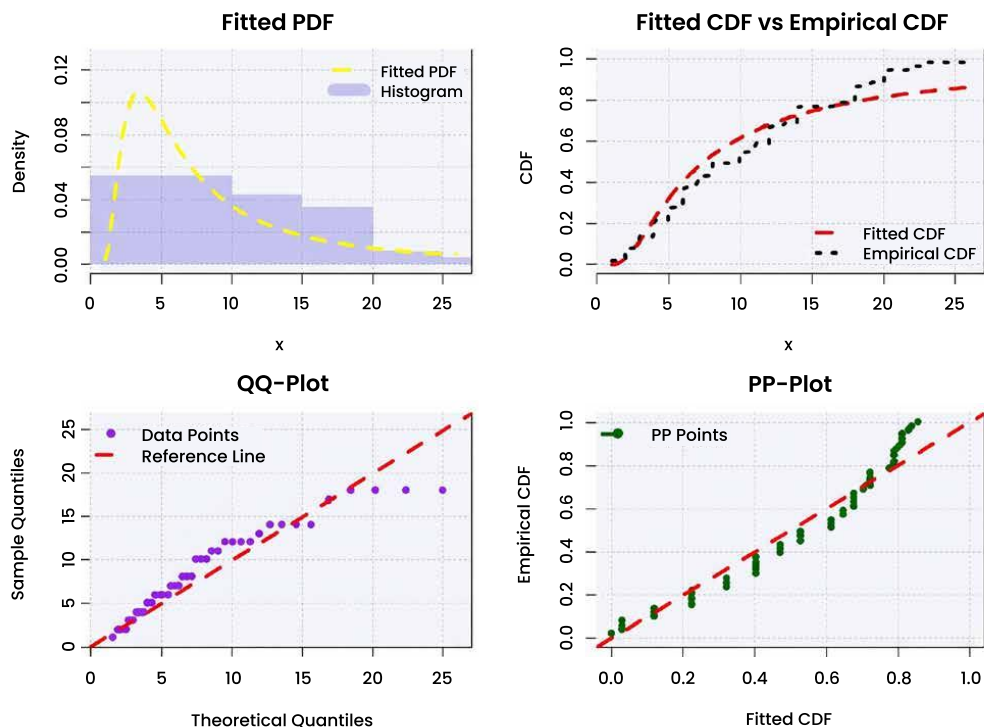


Fig. (13). Fitted plots for average maximum temperature of the environmental data.

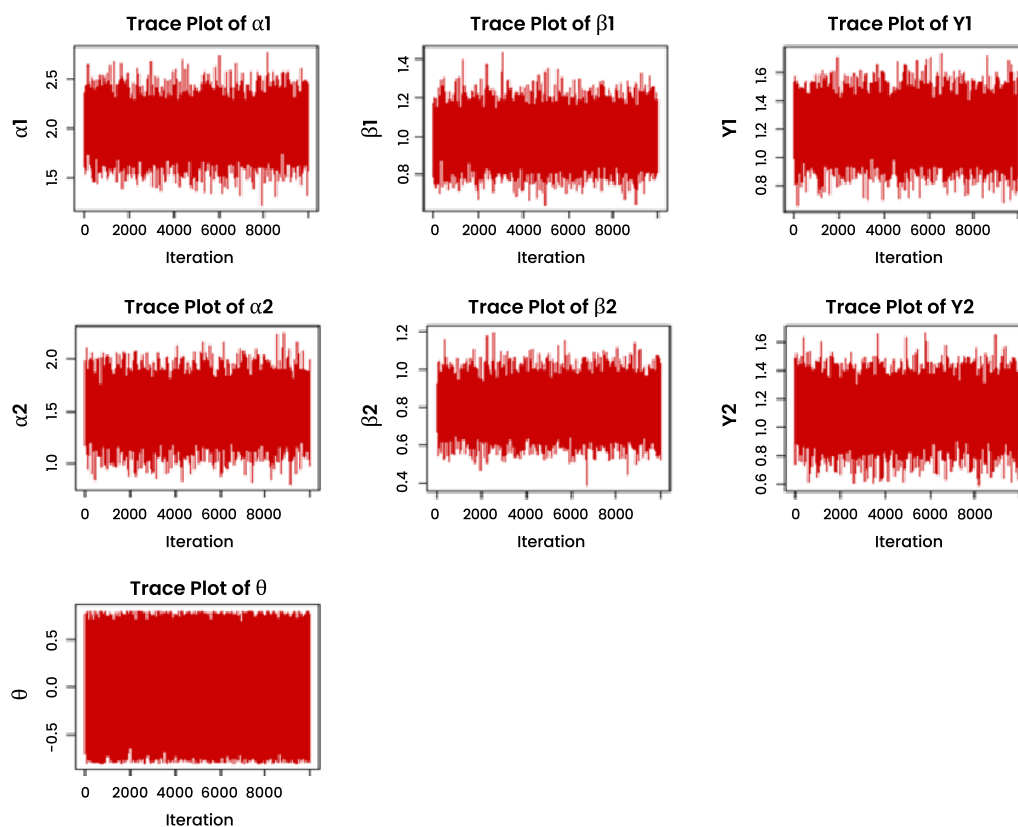


Fig. (14). Trace plot MCMC results for economic data.

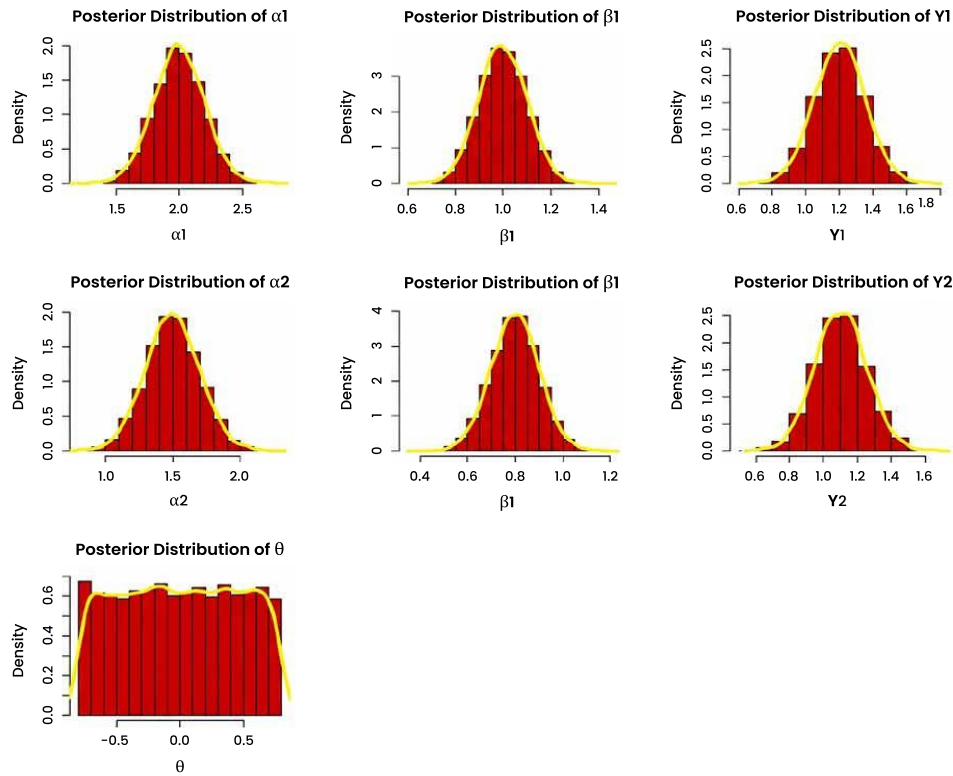


Fig. (15). Histogram plot with density posterior curve of MCMC results for economic data.

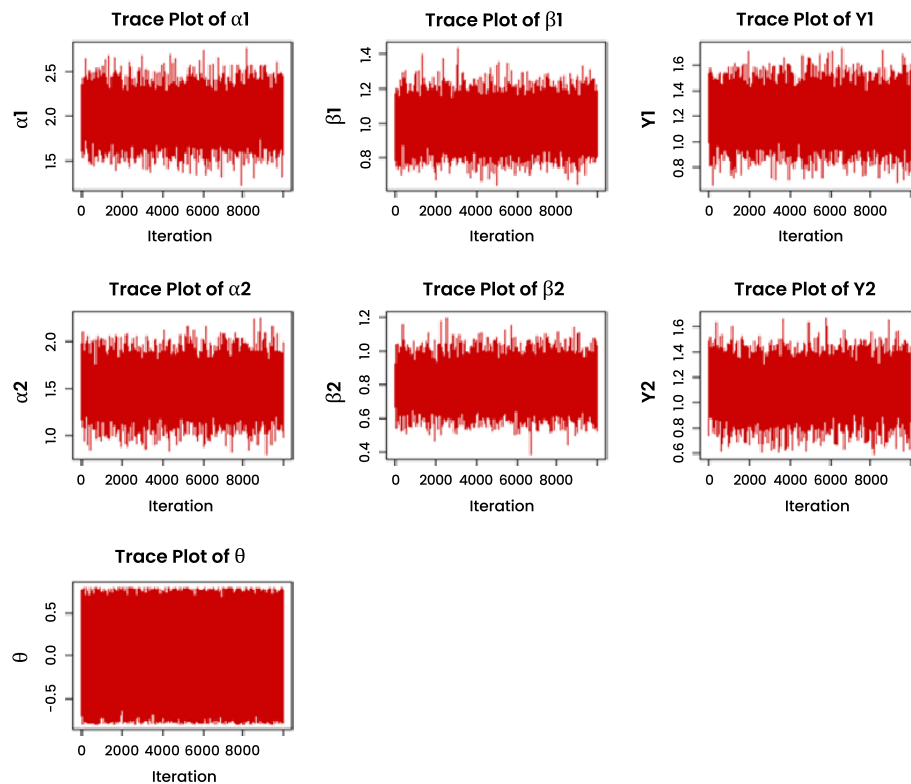


Fig. (16). Trace plot MCMC results for enviornmental data.

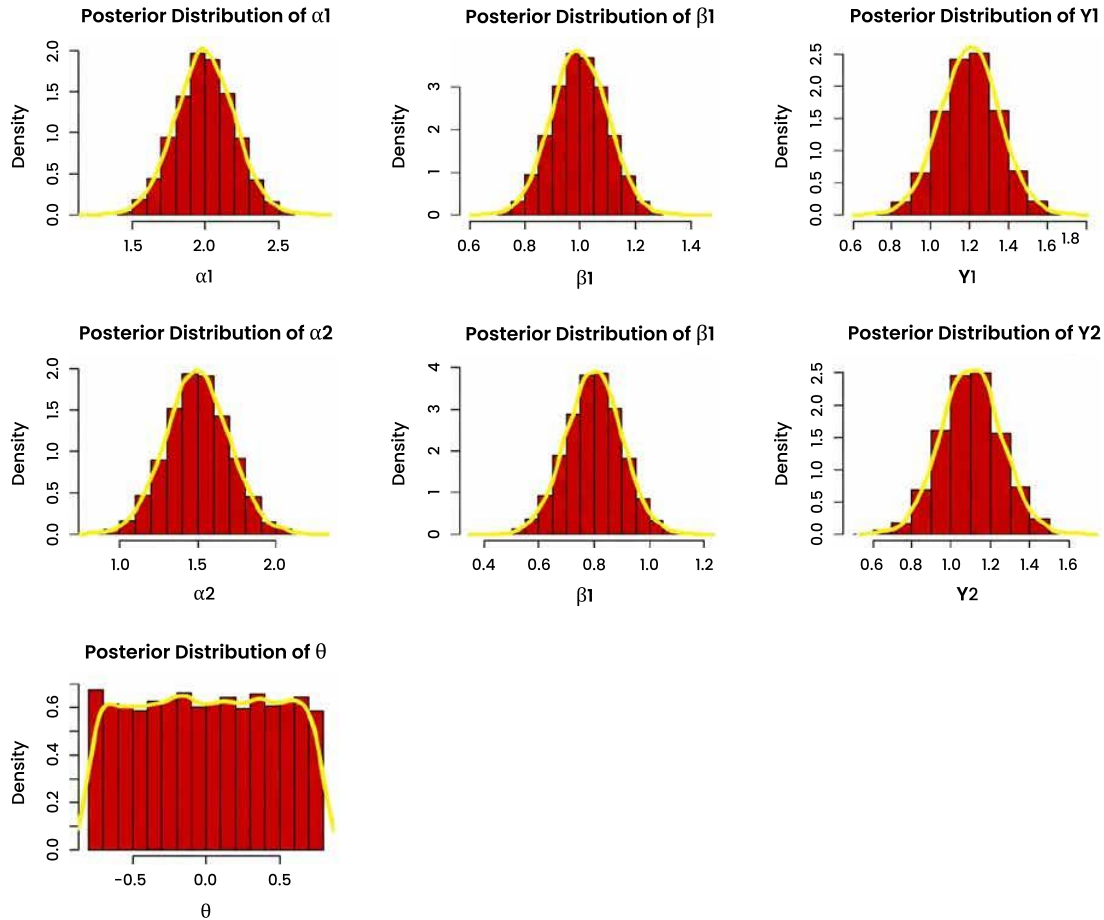


Fig. (17). Histogram plot with density posterior curve of MCMC results for environment data.

LIMITATION

Although the proposed FGM-GIW provides a flexible framework for modeling dependent lifetime data with analytically tractable properties, some limitations should be acknowledged. The dependence structure is governed by the FGM copula, which is computationally convenient and mathematically attractive but has a restricted ability to model strong dependence and tail dependence. Therefore, the proposed model is more suitable for applications involving weak to moderate dependence between variables. Future research may focus on extending this framework by incorporating more flexible copula families, such as Clayton, Frank, Gumbel, or Gaussian copulas, to accommodate a wider range of dependence structures. Furthermore, developing higher-dimensional extensions of the proposed model would be an interesting direction for future investigation.

APPENDIX A

Expressions for fisher information matrix.

$$I_{\alpha_1 \alpha_1} = -\frac{n\beta_1}{\alpha_1^2} - \sum_{i=1}^n \frac{\beta_1 \left(-\frac{(\alpha_1)^{-2+\beta_1}}{x_i} + \frac{(\alpha_1)^{-2+\beta_1} \beta_1}{x_i} \right) \gamma_1}{x_i} + \sum_{i=1}^n \left[\frac{2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \right) \theta \beta_1 \left(-\frac{(\alpha_1)^{-2+\beta_1}}{x_i} + \frac{(\alpha_1)^{-2+\beta_1} \beta_1}{x_i} \right) \gamma_1}{x_i} \right] -$$

$$\begin{aligned}
& \sum_{i=1}^n \frac{2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \theta \left(\frac{\alpha_1}{x_i}\right)^{-2+2\beta_1} \beta_1^2 \gamma_1^2}{x_i^2} \\
I_{\alpha_1 \beta_1} &= \frac{n}{\alpha_1} - \sum_{i=1}^n \left(\frac{\left(\frac{\alpha_1}{x_i}\right)^{-1+\beta_1} \gamma_1}{x_i} + \frac{\log \left[\frac{\alpha_1}{x_i}\right] \left(\frac{\alpha_1}{x_i}\right)^{-1+\beta_1} \beta_1 \gamma_1}{x_i} \right) + \\
& \sum_{i=1}^n \frac{2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \theta \left(\frac{\alpha_1}{x_i}\right)^{-1+\beta_1} \gamma_1}{x_i} + \sum_{i=1}^n \frac{2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \theta \log \left[\frac{\alpha_1}{x_i}\right] \left(\frac{\alpha_1}{x_i}\right)^{-1+\beta_1} \beta_1 \gamma_1}{x_i} \\
& - \sum_{i=1}^n \frac{2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \theta \log \left[\frac{\alpha_1}{x_i}\right] \left(\frac{\alpha_1}{x_i}\right)^{-1+2\beta_1} \beta_1 \gamma_1^2}{x_i} \\
I_{\alpha_1 \gamma_1} &= - \sum_{i=1}^n \frac{\left(\frac{\alpha_1}{x_i}\right)^{-1+\beta_1} \beta_1}{x_i} \\
& + \sum_{i=1}^n \left(\frac{2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \theta \left(\frac{\alpha_1}{x_i}\right)^{-1+\beta_1} \beta_1}{x_i} - \frac{2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \theta \left(\frac{\alpha_1}{x_i}\right)^{-1+2\beta_1} \beta_1 \gamma_1}{x_i} \right) \\
I_{\alpha_1 \theta} &= \sum_{i=1}^n \frac{2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \left(\frac{\alpha_1}{x_i}\right)^{-1+\beta_1} \beta_1 \gamma_1}{x_i} \\
I_{\alpha_1 \beta_1} &= \frac{n}{\alpha_1} - \sum_{i=1}^n \left(\frac{\left(\frac{\alpha_1}{x_i}\right)^{-1+\beta_1} \gamma_1}{x_i} + \frac{\log \left[\frac{\alpha_1}{x_i}\right] \left(\frac{\alpha_1}{x_i}\right)^{-1+\beta_1} \beta_1 \gamma_1}{x_i} \right) + \\
& \sum_{i=1}^n \frac{2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \theta \left(\frac{\alpha_1}{x_i}\right)^{-1+\beta_1} \gamma_1}{x_i} + \sum_{i=1}^n \frac{2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \theta \log \left[\frac{\alpha_1}{x_i}\right] \left(\frac{\alpha_1}{x_i}\right)^{-1+\beta_1} \beta_1 \gamma_1}{x_i} - \\
& \sum_{i=1}^n \frac{2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \theta \log \left[\frac{\alpha_1}{x_i}\right] \left(\frac{\alpha_1}{x_i}\right)^{-1+2\beta_1} \beta_1 \gamma_1^2}{x_i}
\end{aligned}$$

$$\begin{aligned}
I_{\beta_1\beta_1} &= -\frac{n}{\beta_1^2} + \sum_{i=1}^n \log[x_i]^2 x_i^{-1-\beta_1} y_i^{-1-\beta_2} - \sum_{i=1}^n \log\left[\frac{\alpha_1}{x_i}\right]^2 \left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1 + \\
\sum_{i=1}^n &\left(2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \theta \log\left[\frac{\alpha_1}{x_i}\right]^2 \left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \theta \log\left[\frac{\alpha_1}{x_i}\right]^2 \left(\frac{\alpha_1}{x_i}\right)^{2\beta_1} \gamma_1^2\right) \\
I_{\beta_1\gamma_1} &= -\sum_{i=1}^n \log\left[\frac{\alpha_1}{x_i}\right] \left(\frac{\alpha_1}{x_i}\right)^{\beta_1} + \\
\sum_{i=1}^n &\left(2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \theta \log\left[\frac{\alpha_1}{x_i}\right] \left(\frac{\alpha_1}{x_i}\right)^{\beta_1} - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \theta \log\left[\frac{\alpha_1}{x_i}\right] \left(\frac{\alpha_1}{x_i}\right)^{2\beta_1} \gamma_1\right) \\
I_{\beta_1\theta} &= \sum_{i=1}^n 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \log\left[\frac{\alpha_1}{x_i}\right] \left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1 \\
I_{\alpha_1\gamma_1} &= -\sum_{i=1}^n \frac{\left(\frac{\alpha_1}{x_i}\right)^{-1+\beta_1} \beta_1}{x_i} + \\
\sum_{i=1}^n &\left(\frac{2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \theta \left(\frac{\alpha_1}{x_i}\right)^{-1+\beta_1} \beta_1}{x_i} - \frac{2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \theta \left(\frac{\alpha_1}{x_i}\right)^{-1+2\beta_1} \beta_1 \gamma_1}{x_i}\right) \\
I_{\beta_1\gamma_1} &= -\sum_{i=1}^n \log\left[\frac{\alpha_1}{x_i}\right] \left(\frac{\alpha_1}{x_i}\right)^{\beta_1} + \\
\sum_{i=1}^n &\left(2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \theta \log\left[\frac{\alpha_1}{x_i}\right] \left(\frac{\alpha_1}{x_i}\right)^{\beta_1} - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \theta \log\left[\frac{\alpha_1}{x_i}\right] \left(\frac{\alpha_1}{x_i}\right)^{2\beta_1} \gamma_1\right) \\
I_{\gamma_1\gamma_1} &= -\frac{n}{\gamma_1^2} + \sum_{i=1}^n -2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \theta \left(\frac{\alpha_1}{x_i}\right)^{2\beta_1} \\
I_{\gamma_1\theta} &= \sum_{i=1}^n 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \\
I_{\alpha_2\alpha_2} &= -\frac{n\beta_2}{\alpha_2^2} - \sum_{i=1}^n \frac{\beta_2 \left(-\frac{\left(\frac{\alpha_2}{y_i}\right)^{-2+\beta_2}}{y_i} + \frac{\left(\frac{\alpha_2}{y_i}\right)^{-2+\beta_2} \beta_2}{y_i}\right) \gamma_2}{y_i} + \\
\sum_{i=1}^n &\frac{2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \theta \beta_2 \left(-\frac{\left(\frac{\alpha_2}{y_i}\right)^{-2+\beta_2}}{y_i} + \frac{\left(\frac{\alpha_2}{y_i}\right)^{-2+\beta_2} \beta_2}{y_i}\right) \gamma_2}{y_i}
\end{aligned}$$

$$\begin{aligned}
& \sum_{i=1}^n \frac{2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \theta \left(\frac{\alpha_2}{y_i}\right)^{-2+2\beta_2} \beta_2^2 \gamma_2^2}{y_i^2} \\
I_{\alpha_2 \beta_2} &= \frac{n}{\alpha_2} - \sum_{i=1}^n \left(\frac{\left(\frac{\alpha_2}{y_i}\right)^{-1+\beta_2} \gamma_2}{y_i} + \frac{\log \left[\frac{\alpha_2}{y_i}\right] \left(\frac{\alpha_2}{y_i}\right)^{-1+\beta_2} \beta_2 \gamma_2}{y_i} \right) + \\
& \sum_{i=1}^n \frac{2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \theta \left(\frac{\alpha_2}{y_i}\right)^{-1+\beta_2} \gamma_2}{y_i} + \sum_{i=1}^n \frac{2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \theta \log \left[\frac{\alpha_2}{y_i}\right] \left(\frac{\alpha_2}{y_i}\right)^{-1+\beta_2} \beta_2 \gamma_2}{y_i} - \\
& \sum_{i=1}^n \frac{2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \theta \log \left[\frac{\alpha_2}{y_i}\right] \left(\frac{\alpha_2}{y_i}\right)^{-1+2\beta_2} \beta_2 \gamma_2^2}{y_i} \\
I_{\alpha_2 \gamma_2} &= - \sum_{i=1}^n \frac{\left(\frac{\alpha_2}{y_i}\right)^{-1+\beta_2} \beta_2}{y_i} + \\
& \sum_{i=1}^n \left(\frac{2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \theta \left(\frac{\alpha_2}{y_i}\right)^{-1+\beta_2} \beta_2}{y_i} - \frac{2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \theta \left(\frac{\alpha_2}{y_i}\right)^{-1+2\beta_2} \beta_2 \gamma_2}{y_i} \right) \\
I_{\alpha_2 \gamma_2} &= - \sum_{i=1}^n \frac{\left(\frac{\alpha_2}{y_i}\right)^{-1+\beta_2} \beta_2}{y_i} + \\
& \sum_{i=1}^n \left(\frac{2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \theta \left(\frac{\alpha_2}{y_i}\right)^{-1+\beta_2} \beta_2}{y_i} - \frac{2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \theta \left(\frac{\alpha_2}{y_i}\right)^{-1+2\beta_2} \beta_2 \gamma_2}{y_i} \right) \\
I_{\alpha_2 \theta} &= \sum_{i=1}^n \frac{2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \left(\frac{\alpha_2}{y_i}\right)^{-1+\beta_2} \beta_2 \gamma_2}{y_i} \\
I_{\alpha_2 \beta_2} &= \frac{n}{\alpha_2} - \sum_{i=1}^n \left(\frac{\left(\frac{\alpha_2}{y_i}\right)^{-1+\beta_2} \gamma_2}{y_i} + \frac{\log \left[\frac{\alpha_2}{y_i}\right] \left(\frac{\alpha_2}{y_i}\right)^{-1+\beta_2} \beta_2 \gamma_2}{y_i} \right) + \\
& \sum_{i=1}^n \frac{2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \theta \left(\frac{\alpha_2}{y_i}\right)^{-1+\beta_2} \gamma_2}{y_i} + \sum_{i=1}^n \frac{2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \theta \log \left[\frac{\alpha_2}{y_i}\right] \left(\frac{\alpha_2}{y_i}\right)^{-1+\beta_2} \beta_2 \gamma_2}{y_i}
\end{aligned}$$

$$\begin{aligned}
& \sum_{i=1}^n \frac{2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \theta \log \left[\frac{\alpha_2}{y_i}\right] \left(\frac{\alpha_2}{y_i}\right)^{-1+2\beta_2} \beta_2 \gamma_2^2}{y_i} \\
I_{\beta_2 \beta_2} &= -\frac{n}{\beta_2^2} + \sum_{i=1}^n \log[y_i]^2 x_i^{-1-\beta_1} y_i^{-1-\beta_2} - \sum_{i=1}^n \log \left[\frac{\alpha_2}{y_i}\right]^2 \left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2 + \\
& \sum_{i=1}^n \left(2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \theta \log \left[\frac{\alpha_2}{y_i}\right]^2 \left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \theta \log \left[\frac{\alpha_2}{y_i}\right]^2 \left(\frac{\alpha_2}{y_i}\right)^{2\beta_2} \gamma_2^2 \right) \\
I_{\beta_2 \gamma_2} &= -\sum_{i=1}^n \log \left[\frac{\alpha_2}{y_i}\right] \left(\frac{\alpha_2}{y_i}\right)^{\beta_2} + \\
& \sum_{i=1}^n \left(2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \theta \log \left[\frac{\alpha_2}{y_i}\right] \left(\frac{\alpha_2}{y_i}\right)^{\beta_2} - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \theta \log \left[\frac{\alpha_2}{y_i}\right] \left(\frac{\alpha_2}{y_i}\right)^{2\beta_2} \gamma_2 \right) \\
I_{\beta_2 \theta} &= \sum_{i=1}^n 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \log \left[\frac{\alpha_2}{y_i}\right] \left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2 \\
I_{\alpha_2 \gamma_2} &= -\sum_{i=1}^n \frac{\left(\frac{\alpha_2}{y_i}\right)^{-1+\beta_2} \beta_2}{y_i} + \\
& \sum_{i=1}^n \left(\frac{2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \theta \left(\frac{\alpha_2}{y_i}\right)^{-1+\beta_2} \beta_2}{y_i} - \frac{2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \theta \left(\frac{\alpha_2}{y_i}\right)^{-1+2\beta_2} \beta_2 \gamma_2}{y_i} \right) \\
I_{\beta_2 \gamma_2} &= -\sum_{i=1}^n \log \left[\frac{\alpha_2}{y_i}\right] \left(\frac{\alpha_2}{y_i}\right)^{\beta_2} + \\
& \sum_{i=1}^n \left(2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \theta \log \left[\frac{\alpha_2}{y_i}\right] \left(\frac{\alpha_2}{y_i}\right)^{\beta_2} - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \theta \log \left[\frac{\alpha_2}{y_i}\right] \left(\frac{\alpha_2}{y_i}\right)^{2\beta_2} \gamma_2 \right) \\
I_{\gamma_2 \gamma_2} &= -\frac{n}{\gamma_2^2} + \sum_{i=1}^n -2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \theta \left(\frac{\alpha_2}{y_i}\right)^{2\beta_2} \\
I_{\gamma_2 \theta} &= \sum_{i=1}^n 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \\
I_{\alpha_1 \theta} &= \sum_{i=1}^n \frac{2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \left(\frac{\alpha_1}{x_i}\right)^{-1+\beta_1} \beta_1 \gamma_1}{x_i}
\end{aligned}$$

$$\begin{aligned}
I_{\beta_1\theta} &= \sum_{i=1}^n 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \log \left[\frac{\alpha_1}{x_i}\right] \left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1 \\
I_{\gamma_1\theta} &= \sum_{i=1}^n 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1} \left(1 - 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2}\right) \left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \\
I_{\alpha_2\theta} &= \sum_{i=1}^n \frac{2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \left(\frac{\alpha_2}{y_i}\right)^{-1+\beta_2} \beta_2 \gamma_2}{y_i} \\
I_{\beta_2\theta} &= \sum_{i=1}^n 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \log \left[\frac{\alpha_2}{y_i}\right] \left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2 \\
I_{\gamma_2\theta} &= \sum_{i=1}^n 2e^{-\left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \gamma_2} \left(1 - 2e^{-\left(\frac{\alpha_1}{x_i}\right)^{\beta_1} \gamma_1}\right) \left(\frac{\alpha_2}{y_i}\right)^{\beta_2} \\
I_{\theta\theta} &= 0
\end{aligned}$$

‘Python’ code for plotting the CDF and PDF of univariate Generalized Inverse Weibull distribution.

```
import fitPlotpy as fitplt
```

```
import numpy as np
```

```
# CDF of Generalized Inverse Weibull distribution
```

```
def cdf_GIWD(x, α, β, γ):
```

```
    return np.exp(-γ * (α / x)**(β))
```

```
# Parameter values
```

```
param = [
```

```
    {"α": 2, "β": 5, "γ": 0.5},
```

```
    {"α": 3, "β": 5, "γ": 0.5},
```

```
    {"α": 4, "β": 5, "γ": 0.5},
```

```
    {"α": 5, "β": 5, "γ": 0.5},
```

```
    {"α": 7, "β": 5, "γ": 0.5},
```

```
]
```

```
# Plot the CDF
```

```
fitplt.plot_cdf(cdf=cdf_GIWD, param_list=param)
```

```
# PDF of Generalized Inverse Weibull distribution
```

```
def pdf_GIWD(x, α, β, γ):
```

```
    return γ * β * α**(β) * x**(-β-1) * np.exp(-γ * (α / x)**(β))
```

```
# Parameter values
```

```
param = [
```

```

{"α": 2, "β": 5, "γ": 0.5},
{"α": 3, "β": 5, "γ": 0.5},
{"α": 4, "β": 5, "γ": 0.5},
{"α": 5, "β": 5, "γ": 0.5},
{"α": 7, "β": 5, "γ": 0.5},
]
# Plot the PDF
fitplt.plot_pdf(pdf=pdf_GIWD, param_list=param)

```

LIST OF ABBREVIATIONS

AIC	= Akaike Information Criteria
BIC	= Bayesian Information Criteria
CDF	= Cumulative Distribution Function
CAIC	= Corrected Akaike Information Criteria
FGM	= Farlie Gumbel Morgenstern
FGM BXD	= FGM Burr XII Distribution
FGM-GIW	= FGM Bivariate Generalized Inverse Weibull
FGM LD	= FGM Lomax Distribution
FGM WD	= FGM Weibull Distribution
FGM ILD	= FGM Inverse Lomax Distribution
FGM IWD	= FGM Inverse Weibull Distribution
HPD	= Highest Posterior Density
HQIC	= Hannan Quinn Information Criteria
IFM	= Inference Functions for Margins
MLE	= Maximum Likelihood Estimation
MCMC	= Markov Chain Monte Carlo
PDF	= Probability Density Function

AUTHORS' CONTRIBUTIONS

Conceptualization, M.O.; Methodology, M.O. and S.M.Z.; Software, M.O., H.H., S.M.Z., and M.H.U.; Validation, M.O. and M.H.U.; Formal Analysis, M.O. and S.M.Z.; Investigation, M.O., H.H., and M.H.U.; Data Curation, M.O. and H.H.; Resources, S.M.Z. and M.H.U.; Writing-original draft preparation, M.O.; Writing-review and editing, M.O., S.M.Z., and M.H.U.; Visualization, M.O.; Supervision, M.O.; Project Administration, S.M.Z. and M.H.U. All authors contributed to the interpretation and discussion of the results, critically reviewed the manuscript for important intellectual content, and approved the final version of the manuscript. All authors have read and agreed to the published version of the manuscript.

ETHICAL APPROVAL & INFORMED CONSENT

Ethical approval and informed consent were not required for this study because the research used publicly available data.

AVAILABILITY OF DATA AND MATERIALS

The datasets used in this study are publicly available. The economic dataset consists of annual observations from 1980 to 2010 on exports of goods and services and GDP growth, as reported in Table 6 and obtained from the source cited in the manuscript [30]. The environmental dataset consists of observations for 51 major cities in the United States, including average precipitation and average maximum temperature. These data were compiled by the National Climatic Data Center (NCDC) and are publicly accessible through the National Centers for Environmental Information (NCEI) website: <https://www.ncei.noaa.gov/>.

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CONFLICT OF INTEREST

The authors declare that they have no competing interests that could have influenced the conduct or findings of this study.

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DECLARATION OF AI

During the preparation of this manuscript, Artificial Intelligence (AI)-based tools were used for language improvement, grammar checking, and refinement of the presentation of the manuscript. The authors take full responsibility for the content, accuracy, interpretation, and conclusions presented in the manuscript. AI tools were not used to generate or fabricate research data, results, or scientific conclusions.

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